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Abstract: The grade filtration of a finitely generated left module M over an Auslander regular ring D is a built-in classification of the elements of M in terms of their grades (or their (co)dimensions if D is also a Cohen-Macaulay ring). In this paper, we show how grade filtration can be explicitly characterized by means of elementary methods of homological algebra. Our approach avoids the use of sophisticated methods such as bidualizing complexes, spectral sequences, associated cohomology, and Spencer cohomology used in the literature of algebraic analysis. Efficient implementations dedicated to the computation of grade filtration can then be easily developed in the standard computer algebra systems (see the Maple package PurityFiltration and the GAP4 package `AbelianSystems`). Moreover, this characterization of grade filtration is shown to induce a new presentation of the left D-module M which is defined by a block-triangular matrix formed by equidimensional diagonal blocks. The linear functional system associated with the left D-module M can then be integrated in cascade by successively solving inhomogeneous linear functional systems defined by equidimensional homogeneous linear systems of increasing dimension. This equivalent linear system generally simplifies the computation of closed-form solutions of the original linear system. In particular, many classes of underdetermined/overdetermined linear systems of partial differential equations can be explicitly integrated by the packages PurityFiltration and `AbelianSystems`, but not by computer algebra systems such as Maple.

Key-words: Algebraic analysis, grade filtration, module theory, homological algebra, symbolic computation, mathematical systems theory, underdetermined/overdetermined linear functional systems, linear systems of partial differential equations.

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Filtration par grade des systèmes linéaires fonctionnels

Résumé : La filtration par grade d'un module à gauche M finiment engendré sur un anneau Auslander-régulier D est une classification intrinsèque des éléments de M en fonction de leurs grades (ou de leurs (co)dimensions si D est aussi un anneau de Cohen-Macaulay). Dans ce papier, nous montrons comment la filtration par grade peut être explicitement caractérisée au moyen de techniques élémentaires d'algèbre homologique. Notre approche évite l'utilisation de techniques sophistiquées telles que les complexes bidualisants, les suites spectrales, la cohomologie associée et la cohomologie de Spencer utilisées dans la littérature d'analyse algébrique. Des implantations efficaces dédiées au calcul de la filtration par grade peuvent alors être facilement développées dans les systèmes standards de calcul formel (voir le package PurityFiltration de Maple et le package `AbelianSystems` de GAP4). De plus, cette caractérisation de la filtration par grade induit une nouvelle présentation du D-module à gauche M qui est définie par une matrice triangulaire par blocs formée de blocs diagonaux équidimensionnels. Le système linéaire fonctionnel associé au D-module à gauche M peut alors être intégré en cascade par la résolution successive de systèmes linéaires fonctionnels inhomogènes définis par des systèmes linéaires homogènes équidimensionnels de dimension croissante. Ce système linéaire équivalent simplifie généralement le calcul des solutions sous formes closes du système linéaire originel. En particulier, de nombreux systèmes linéaires sur-déterminés/sous-déterminés d'équations aux dérivées partielles peuvent être explicitement intégrés au moyen des packages PurityFiltration et `AbelianSystems`, alors qu'ils ne peuvent l'être par des systèmes de calcul formel tels que Maple.

Mots-clés : Analyse algébrique, filtration par grade, théorie des modules, algèbre homologique, calcul formel, théorie mathématique des systèmes, systèmes linéaires fonctionnels sur-déterminés/sous-déterminés, systèmes linéaires d'équations aux dérivées partielles.

1 Introduction

The theory of *linear functional systems* such as linear systems of partial differential/time-delay/difference equations is a rich branch of mathematics which finds its foundation in mathematical physics. Different analytic methods can be used to study *determined* linear functional systems (see, e.g., [19]), namely linear functional systems containing as many unknown functions as functionally independent linear equations. *Overdetermined* (resp., *underdetermined*) linear functional systems, namely linear functional systems containing fewer (resp., more) unknown functions than functionally independent linear equations, also find important applications in mathematical physics (see, e.g., [13, 38]), in differential geometry (see, e.g., [24, 38]), or in mathematical systems theory (see, e.g., [14, 36, 38, 40]). Formal methods for studying overdetermined linear systems of PD equations can be traced back to the works of Cartan, Riquier and Janet [27]. A modern approach was developed in the sixties by Spencer and his collaborators (see, e.g., [38, 51]). *Gröbner bases* and *Janet bases* [12, 27] over a noncommutative polynomial ring of functional operators are nowadays two fundamental computational tools for the formal study of overdetermined linear functional systems (see, e.g., [14, 31, 48]).

Despite these important computational methods, computer algebra systems still have many difficulties to find closed-form solutions of overdetermined or undetermined linear functional systems (when they exist), for instance of linear systems of PD equations. One of the main reasons for this failure is that linear functional systems generally mix together unknown functions which satisfy linear functional systems of different dimension. For instance, the integration of the unknown functions of an overdetermined linear systems of PD equations depends on arbitrary functions of a certain number of the independent variables (due to the *Cartan-Kähler-Janet theorem* which generalizes the well-known *Cauchy-Kowalevski theorem*) (see, e.g., [27, 38, 51]). The maximal number of independent variables which appear in these arbitrary functions (sometimes plus the number of independent variables) is called the *dimension* of the system. Hence, an important issue for the study of overdetermined linear functional systems is to determine the unknown functions or their linear functional combinations which satisfy a linear functional system of a given dimension. This problem, related to the *equidimensional decomposition* of algebraic varieties (see, e.g., [20, 25, 49]), has lengthly been studied within *algebraic analysis* and *algebraic/analytic D-module theory* [9, 10, 11, 33] by Roos [49], Sato and Kashiwara [29, 30], Björk [9, 10], Ginsburg [23], and others. This problem corresponds to the so-called *grade filtration* $\{M_i\}_{i \geq 0}$ (also called *bidualizing* or *purity filtration*) of the finitely generated left D-module M which defines the linear system of PD equations, where D is a noncommutative polynomial ring of PD operators satisfying certain regularity conditions (e.g., D is an *Auslander regular ring*). This filtration of M is defined by the left D-submodules M_i's of M formed by the elements of M having a *codimension* (or a *grade*) greater or equal to i. The existence of the grade filtration of a finitely generated left/right module M over an Auslander regular ring D is proved in [9, 10, 23, 32, 49] (resp., in [30, 29]) using *bidualizing complexes* and *spectral sequence* arguments (resp., *derived categories*, *derived functors* and *associated cohomology* [25]), i.e., by means of sophisticated homological algebra techniques (resp., modern developments of category theory). See also [38, 39] (resp., [37]) for a recent study of grade filtration based on *Spencer cohomology* and *Spencer sequences* (resp., *Gabriel localization* for commutative polynomial rings). Despite the difficulties for the computation of the spectral sequences defining the grade filtration, they were recently made constructive in [2, 3] thanks to the new concept of *generalized morphisms*, and they were implemented in the `homalg` package [8] of the system GAP4 [22] (`homalg` is a package dedicated to homological algebra oriented computations). To our knowledge, it is the first implementation of the computation of the grade filtration in a computer algebra system.

We refer the reader to [20, 25, 49] (resp., [9, 10, 23, 29]) for applications of grade filtration to algebraic geometry (resp., algebraic analysis). Finally, techniques based on grade filtration have recently been introduced in mathematical systems theory (see [4, 37, 38, 39, 40, 41, 42, 43, 44]).

The purpose of this paper is to develop a new algorithm which computes the grade filtration of a finitely generated left module M over a *noetherian regular domain* D satisfying a slightly weaker condition than the standard *Auslander condition* (see, e.g., [9, 10]). In particular, many important classes of noncommutative polynomial rings of functional systems satisfy these conditions. The first benefit of this new algorithm is that it is an extension of the methods developed in [1, 14, 30, 38, 40] for the classification of modules (torsion modules, modules with torsion submodules, torsion-free/reflexive/projective modules). These methods have recently been applied to solve the problem of parametrizing underdetermined linear functional systems by means of arbitrary functions (*potentials*) studied in mathematical physics and in control theory (see [14, 15, 21, 38, 40, 53]). The second benefit of this algorithm is that it is conceptually much simpler than the algorithms based on bidualizing complexes, spectral sequences and associated cohomology. In particular, it can be easily implemented in any computer algebra system in which Gröbner basis techniques are available (e.g., Maple, Mathematica, Singular, Macaulay2, Magma). The corresponding algorithm was implemented by the author in the Maple package PurityFiltration [45] built upon OreModules [15]. Using the PurityFiltration package, classes of overdetermined/underdetermined linear systems of PD equations which cannot be directly integrated by Maple can be explicitly solved [45] (see also the forthcoming `homalg` based package `D-modules`). Moreover, the algorithm has also been implemented recently in the `homalg` project package `AbelianSystems` [7] developed in collaboration with M. Barakat (University of Kaiserslautern). This implementation is much faster than the original `homalg` command based on spectral sequence computation (10 times faster on small PD examples), and thus it can be used to study larger examples. We hope that the results developed in this paper and demonstrated by the PurityFiltration and `AbelianSystems` packages will be used in the future to improve standard computer algebra systems such as Maple or Mathematica for the symbolic integration of overdetermined/underdetermined linear functional systems. More generally, this new algorithm holds for *constructive abelian categories* [6], and thus it can be used in different contexts such as the computation of the grade filtration of coherent sheaves over projective schemes as shown in the `homalg` project package `Sheaves` [5].

Since techniques of module theory, homological algebra and algebraic analysis are not largely well-known, they are summarized in Section 2. The main results about grade filtration are developed in Section 3. In Section 4, we show how the concept of grade filtration can be used to compute an equivalent block-triangular form of a linear functional system whose diagonal blocks define equidimensional linear functional systems. The integration of the original system is then equivalent to a cascade integration of inhomogeneous linear functional systems, the corresponding homogeneous linear systems being equidimensional and of increasing dimension (e.g., we first integrate a 0-dimensional/*holonomic* homogenous linear system, then an inhomogeneous linear systems defined by a 1-dimensional/*subholonomic* homogeneous linear system, ...). In Section 5, we briefly give a few extensions of the results obtained in Section 3. Finally, in Appendix, we demonstrate the PurityFiltration package through different explicit examples.

2 Algebraic analysis approach to linear functional systems

In what follows, D will always be a *noetherian ring*, i.e., a ring D that is both a left and a right noetherian ring (see, e.g., [50]). Moreover, the set of $q \times p$ matrices with entries in D is denoted by $D^{q\times p}$ and the unit of the ring $D^{p\times p}$ by I_p. If $\mathcal{F}$ is a left D-module (e.g., $\mathcal{F}=D$) and $R\in D^{q\times p}$, then $.R$ and $R.$ are respectively the left D-homomorphism (i.e., the left D-linear map) and the abelian group homomorphism (i.e., $\mathbb{Z}$-homomorphism) defined by:

$$\begin{array}{rcl} .R\colon D^{1\times q} & \longrightarrow & D^{1\times p} \\ \lambda=(\lambda_1\ \ldots\ \lambda_q) & \longmapsto & \lambda\, R, \end{array} \qquad \begin{array}{rcl} R.\colon \mathcal{F}^p & \longrightarrow & \mathcal{F}^q \\ \eta=(\eta_1\ \ldots\ \eta_p)^T & \longmapsto & R\,\eta. \end{array}$$

With the above notations, we call *linear system* an abelian group of the form:

$$\ker_{\mathcal{F}}(R.)=\{\eta\in\mathcal{F}^p \mid R\,\eta=0\}.$$

The study of $\ker_{\mathcal{F}}(R.)$ in terms of the *finitely presented left D-module* $M=D^{1\times p}/(D^{1\times q}\,R)$ and of the left D-module $\mathcal{F}$ was first developed in [34]. This idea is nowadays the cornerstone of the *algebraic D-module theory* (or *algebraic analysis*), developed by Bernstein and Sato's school (particularly by Kashiwara), in which D stands for a noncommutative ring of partial differential (PD) operators with coefficients in a *differential ring* (see, e.g., [9, 10, 11, 30, 33]). More precisely, if A is a ring and $\{\delta_i\}_{i=1,\ldots,n}$ are n *commuting derivations* of A, namely, $\delta_i\colon A\longrightarrow A$ satisfies $\delta_i(a_1+a_2)=\delta_i(a_1)+\delta_i(a_2)$, $\delta_i(a_1\,a_2)=\delta_i(a_1)\,a_2+a_1\,\delta_i(a_2)$ for all $a_1,\,a_2\in A$ and for all $i=1,\ldots,n$, and $\delta_i\circ\delta_j=\delta_j\circ\delta_i$ for all $i,j=1,\ldots,n$, then the ring $D=A\langle\partial_1,\ldots,\partial_n\rangle$ of PD operators with coefficients in A is the noncommutative polynomial ring in $\partial_1,\ldots,\partial_n$ which satisfies the relations $\partial_i\,a=a\,\partial_i+\delta_i(a)$ for all $a\in A$ and for all $i=1,\ldots,n$, and $\partial_i\,\partial_j=\partial_j\,\partial_i$ for all $i,j=1,\ldots,n$. Prototype examples of a ring D of PD operators are the so-called *Weyl algebras* $A_n(k)$ and $B_n(k)$ of PD operators with respectively coefficients in $A=k[x_1,\ldots,x_n]$ and in $A=k(x_1,\ldots,x_n)$, where k is a field (that we shall suppose to be of characteristic 0), $\hat{\mathcal{D}}_n(k)$, or $\mathcal{D}_n(k')$ the rings of PD operators with coefficients in the ring of formal power series $A=k[\![x_1,\ldots,x_n]\!]$ or in the ring of locally convergent power series $A=k'\{x_1,\ldots,x_n\}$, where $k'=\mathbb{R}$ or $\mathbb{C}$. These rings are noetherian domains (see, e.g., [9, 11, 33]). If D is a ring of PD operators and $\mathcal{F}$ a left D-module (e.g., $\mathcal{F}=A$), then $R\in D^{q\times p}$ is a matrix of PD operators and the linear system $\ker_{\mathcal{F}}(R.)$ is the k-vector space formed by the $\mathcal{F}$-solutions of the linear system of PD equations $R\,\eta=0$. Within algebraic analysis, more general classes of noncommutative polynomial rings of functional operators can be considered such as *Ore algebras* as explained in [14], which allows one to consider a more general class of linear functional systems.

Let us now explain basic ideas of algebraic analysis. Let $\pi\colon D^{1\times p}\longrightarrow M$ be the left D-homomorphism which maps $\lambda\in D^{1\times q}$ to its residue class $\pi(\lambda)\in M$, and $\{f_j\}_{j=1,\ldots,p}$ the *standard basis* of $D^{1\times p}$, namely, f_j is the row vector of length p with 1 at the j^{th} position and 0 elsewhere. Then, $\{y_j=\pi(f_j)\}_{j=1,\ldots,n}$ is a family of generators of M since for every $m\in M$, there exists $\lambda=(\lambda_1\ \ldots\ \lambda_p)\in D^{1\times p}$ such that $m=\pi(\lambda)$, which yields:

$$m=\pi(\lambda)=\pi\left(\sum_{j=1}^{p}\lambda_j\,f_j\right)=\sum_{j=1}^{p}\lambda_j\,\pi(f_j)=\sum_{j=1}^{p}\lambda_j\,y_j.$$

The family of generators $\{y_j\}_{j=1,\ldots,p}$ of M satisfies D-linear relations: if $R_{i\bullet}$ denotes the i^{th} row of R, then $R_{i\bullet}\in D^{1\times q}\,R$, which yields $\pi(R_{i\bullet})=0$, and thus:

$$\forall\ i=1,\ldots,q,\quad \pi(R_{i\bullet})=\pi\left(\sum_{j=1}^{p}R_{ij}\,f_j\right)=\sum_{j=1}^{p}R_{ij}\,\pi(f_j)=\sum_{j=1}^{p}R_{ij}\,y_j=0.$$

If $y=(y_1\ \ldots\ y_p)^T\in M^p$, then the above relations can be rewritten as $R\,y=0$.

Now, if $\mathcal{F}$ is a left D-module, $\hom_D(M, \mathcal{F})$ the abelian group of left D-homomorphisms from M to $\mathcal{F}$, and $\phi \in \hom_D(M, \mathcal{F})$, then $\eta = (\phi(y_1) \ \dots \ \phi(y_p))^T \in \mathcal{F}^p$ and

$$\forall\, i = 1, \dots, q, \quad \sum_{j=1}^{p} R_{ij}\, \eta_j = \sum_{j=1}^{p} R_{ij}\, \phi(y_j) = \phi\left(\sum_{j=1}^{p} R_{ij}\, y_j\right) = \phi(0) = 0,$$

i.e., $\eta \in \ker_{\mathcal{F}}(R.)$. Conversely, if $\eta \in \ker_{\mathcal{F}}(R.)$, then we can define the map $\phi_\eta \colon M \longrightarrow \mathcal{F}$ by $\phi_\eta(\pi(\lambda)) = \lambda\, \eta$ for all $\lambda \in D^{1\times p}$. Indeed, ϕ_η is well-defined: if $\pi(\lambda) = \pi(\lambda')$, then $\lambda = \lambda' + \mu\, R$, for a certain $\mu \in D^{1\times q}$, which yields $\phi_\eta(\pi(\lambda)) = \lambda\, \eta = \lambda'\, \eta + \mu\, R\, \eta = \lambda'\, \eta$. The map ϕ_η is clearly left D-linear and $\phi_\eta(0) = 0$ since $\phi_\eta\left(\sum_{j=1}^{p} R_{ij}\, y_j\right) = \sum_{j=1}^{p} R_{ij}\, \eta_j = 0$ for all $i = 1, \dots, q$, and thus $\phi_\eta \in \hom_D(M, \mathcal{F})$. If we introduce the following abelian group homomorphisms

$$\begin{array}{rccc} \sigma\colon & \ker_{\mathcal{F}}(R.) & \longrightarrow & \hom_D(M, \mathcal{F}) \\ & \eta & \longmapsto & \phi_\eta, \end{array} \qquad \begin{array}{rccc} \chi\colon & \hom_D(M, \mathcal{F}) & \longrightarrow & \ker_{\mathcal{F}}(R.) \\ & \phi & \longmapsto & (\phi(y_1) \ \dots \ \phi(y_p))^T, \end{array}$$

then $\chi \circ \sigma = \mathrm{id}_{\ker_{\mathcal{F}}(R.)}$ since $\phi_\eta(y_j) = \eta_j$ for all $j = 1, \dots, p$, and $\sigma \circ \chi = \mathrm{id}_{\hom_D(M,\mathcal{F})}$ since $(\sigma \circ \chi)(\phi) = \phi_{(\phi(y_1) \,\dots\, \phi(y_p))^T} = \phi$, which shows that $\chi^{-1} = \sigma$, and proves that $\ker_{\mathcal{F}}(R.)$ and $\hom_D(M, \mathcal{F})$ are isomorphic as abelian groups, which is denoted by $\ker_{\mathcal{F}}(R.) \cong \hom_D(M, \mathcal{F})$.

Theorem 1 ([34])**.** *With the previous notations, we have:*

$$\ker_{\mathcal{F}}(R.) \cong \hom_D(M, \mathcal{F}).$$

Theorem 1 shows that the linear system $\ker_{\mathcal{F}}(R.)$ can be intrinsically studied by means of the two left D-modules $M = D^{1\times p}/(D^{1\times q}\, R)$ and $\mathcal{F}$. The matrix R is a particular *finite presentation* of the left D-module M defined up to isomorphism (see, e.g., [50]). Hence, we can study the solution space $\hom_D(M, \mathcal{F})$ independently of the particular embedding of $\ker_{\mathcal{F}}(R.)$ into $\mathcal{F}^p$. A second benefit of Theorem 1 is that the linear system $\ker_{\mathcal{F}}(R.)$ can be studied by means of the properties of the left D-modules M and $\mathcal{F}$.

Definition 1 ([50])**.** Let D be a noetherian ring and M a finitely generated left D-module.

1. M is *free* if there exists $r \in \mathbb{N} = \{0, 1, 2, \dots\}$ such that $M \cong D^{1\times r}$. Then, r is then called the *rank* of M.

2. M is *projective* if there exist $r \in \mathbb{N}$ and a left D-module N such that $M \oplus N \cong D^{1\times r}$, where $\oplus$ denotes the direct sum of left D-modules.

3. M is *reflexive* if the left D-homomorphism $\varepsilon\colon M \longrightarrow \hom_D(\hom_D(M, D), D)$, defined by $\varepsilon(m)(f) = f(m)$ for all $m \in M$ and for all $f \in \hom_D(M, D)$, is an isomorphism.

4. If D is a domain, then M is *torsion-free* if the *torsion left D-submodule* of M defined by

$$t(M) = \{m \in M \mid \exists\, d \in D \setminus \{0\} \colon\, d\, m = 0\}$$

is reduced to 0, i.e., if $t(M) = 0$.

5. If D is a domain, then M is *torsion* if $t(M) = M$, i.e., if every element of M is a torsion element.

Theorem 2 ([50])**.** *A free module is projective, a projective module is reflexive, and a reflexive module is torsion-free.*

In the next sections, we summarize basic homological techniques which will be used to algorithmically test whether or not M admits torsion elements or is torsion-free, reflexive or projective (see Theorem 5 thereafter). These techniques will then be generalized in Section 3 to obtain an explicit characterization of the so-called *grade filtration* of M.

2.1 Basic homological algebra

Let us shortly recall a few definitions of homological algebra (see, e.g., [50]).

Definition 2. 1. A *complex*, denoted by

$$M_\bullet \; \ldots \xrightarrow{d_{i+2}} M_{i+1} \xrightarrow{d_{i+1}} M_i \xrightarrow{d_i} M_{i-1} \xrightarrow{d_{i-1}} \ldots, \tag{1}$$

is a sequence of left (resp., right) D-modules M_i and of left (resp., right) D-homomorphisms $d_i \colon M_i \longrightarrow M_{i-1}$ that satisfy $\operatorname{im} d_{i+1} \subseteq \ker d_i$, i.e., $d_i \circ d_{i+1} = 0$ for all $i \in \mathbb{Z}$.

2. The *defect of exactness* of (1) at M_i is the left (resp., right) D-module defined by:

$$H_i(M_\bullet) \triangleq \ker d_i / \operatorname{im} d_{i+1}.$$

3. The complex (1) is *exact at* M_i if $H_i(M_\bullet) = 0$, i.e., if $\ker\, d_i = \operatorname{im} d_{i+1}$, and *exact* if $\ker\, d_i = \operatorname{im} d_{i+1}$ for all $i \in \mathbb{Z}$. An exact complex is called an *exact sequence*.

4. An exact sequence of the form

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0, \tag{2}$$

i.e., f is injective, $\ker g = \operatorname{im} f$ and g is surjective, is called a *short exact sequence*.

5. A *projective resolution* of a left D-module M is an exact sequence of the form

$$\ldots \xrightarrow{d_4} P_3 \xrightarrow{d_3} P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \longrightarrow 0,$$

where the P_i's are projective left D-modules and $d_i \in \hom_D(P_i, P_{i-1})$ for all $i \in \mathbb{N}$. The smallest $n \in \mathbb{N}$ such that $P_m = 0$ for all $m > n$ is called the *length* of the projective resolution of M. Similarly for right D-modules.

6. A *free resolution* of a left D-module M is an exact sequence of the form

$$\ldots \xrightarrow{.R_3} D^{1\times p_2} \xrightarrow{.R_2} D^{1\times p_1} \xrightarrow{.R_1} D^{1\times p_0} \xrightarrow{\pi} M \longrightarrow 0, \tag{3}$$

where $R_i \in D^{p_i \times p_{i-1}}$ and $.R_i \colon D^{1\times p_i} \longrightarrow D^{1\times p_{i-1}}$ is defined by $(.R_i)(\lambda) = \lambda\, R_i$.

7. A *free resolution* of a right D-module N is an exact sequence of the form

$$0 \longleftarrow N \xleftarrow{\kappa} D^{q_0} \xleftarrow{S_1.} D^{q_1} \xleftarrow{S_2.} D^{q_2} \xleftarrow{S_3.} \ldots, \tag{4}$$

where $S_i \in D^{q_{i-1}\times q_i}$ and $S_i. \colon D^{q_i} \longrightarrow D^{q_{i-1}}$ is defined by $(S_i.)(\eta) = S_i\, \eta$.

Example 1. If D is a noetherian domain and M a finitely generated left D-module, then we have the following short exact sequence of left D-modules:

$$0 \longrightarrow t(M) \xrightarrow{j} M \xrightarrow{\rho} M/t(M) \longrightarrow 0. \tag{5}$$

Remark 1. A module M is not defined by a unique projective/free resolution: *Fitting's lemma* asserts that if $0 \longrightarrow \ker \pi \longrightarrow P \xrightarrow{\pi} M \longrightarrow 0$ and $0 \longrightarrow \ker \pi' \longrightarrow P' \xrightarrow{\pi'} M \longrightarrow 0$ are two exact sequences, where P and P' are projective/free modules, then $\ker \pi \oplus P' \cong \ker \pi' \oplus P$ (see, e.g., [50]). This isomorphism does not generally imply that $\ker \pi \cong \ker \pi'$. We say that $\ker \pi$ depends on M up to a *projective equivalence* (see, e.g., [50]). Similarly, if we consider two finite presentations of M, $D^{1\times p_1} \xrightarrow{.R_1} D^{1\times p_0} \xrightarrow{\pi} M \longrightarrow 0$ and $D^{1\times p_1'} \xrightarrow{.R_1'} D^{1\times p_0'} \xrightarrow{\pi'} M \longrightarrow 0$, then $\ker_D(.R_1) \oplus D^{1\times (p_1'+p_0)} \cong \ker_D(.R_1') \oplus D^{1\times(p_1+p_0')}$. For more details, see, e.g., [50]. For a constructive proof, see [18]. Similar results hold for all the *syzygy modules* $\ker_D(.R_i)$'s of M.

Since D is a noetherian ring, one can easily prove that every finitely generated left (resp. right) D-module M admits a free resolution (see, e.g., [50]). Now, if $\mathcal{F}$ is a left D-module, then using a free resolution (3) of a finitely generated left D-module M, we can define the *extension abelian groups* $\mathrm{ext}^i_D(M,\mathcal{F})$'s for $i \geq 0$ as follows. Up to abelian group isomorphism, they are defined by the defects of exactness of the following complex of abelian groups

$$\ldots \xleftarrow{R_{i+1}.} \mathcal{F}^{p_i} \xleftarrow{R_i.} \mathcal{F}^{p_{i-1}} \xleftarrow{R_{i-1}.} \ldots \xleftarrow{R_3.} \mathcal{F}^{p_2} \xleftarrow{R_2.} \mathcal{F}^{p_1} \xleftarrow{R_1.} \mathcal{F}^{p_0} \longleftarrow 0, \tag{6}$$

where $R_i.\colon \mathcal{F}^{p_{i-1}} \longrightarrow \mathcal{F}^{p_i}$ is defined by $(R_i.)(\eta) = R_i\,\eta$ for all $\eta \in \mathcal{F}^{p_{i-1}}$ and $i \geq 1$, namely:

$$\begin{cases} \mathrm{ext}^0_D(M,\mathcal{F}) \cong \ker_{\mathcal{F}}(R_1.), \\ \mathrm{ext}^i_D(M,\mathcal{F}) \cong \ker_{\mathcal{F}}(R_{i+1}.)/\mathrm{im}_{\mathcal{F}}(R_i.), \quad i \geq 1. \end{cases} \tag{7}$$

Theorem 1 shows that:

$$\mathrm{ext}^0_D(M,\mathcal{F}) = \hom_D(M,\mathcal{F}).$$

See also, e.g., [50]. We say that the complex (6) is obtained by application of the *contravariant left exact functor* $\hom_D(\,\cdot\,,\mathcal{F})$ to the *reduced (truncated) free resolution* of M, namely, to the complex obtained by removing M from the finite free resolution (3) as follows:

$$\ldots \xrightarrow{.R_4} D^{1\times p_3} \xrightarrow{.R_3} D^{1\times p_2} \xrightarrow{.R_2} D^{1\times p_1} \xrightarrow{.R_1} D^{1\times p_0} \longrightarrow 0. \tag{8}$$

A fundamental theorem of homological algebra asserts that the abelian groups $\mathrm{ext}^i_D(M,\mathcal{F})$'s depend only on the left D-modules M and $\mathcal{F}$ (up to abelian group isomorphism), i.e., they do not depend on the choice of the free resolution (3) of M (see, e.g., [50]). The $\mathrm{ext}^i_D(M,\mathcal{F})$'s can also be defined using projective resolutions of M (see, e.g., [50]). But, this approach is generally less constructive than the one based on free resolutions. In what follows, we shall only consider free resolutions and we let the reader reformulate the different results based on projective resolutions.

The idea of replacing a rather complicated left D-module M by the complex (8) formed by the left D-modules $D^{1\times p_i}$'s (free modules) and trivial left D-homomorphisms $.R_i$'s (defined by matrices) is of paramount importance in the theory of *derived category* developed by Grothendieck and Verdier (see, e.g., [25]). In this paper, we shall show how the grade filtration of M, which is difficult to compute directly on M, can be explicitly characterized by many but simple (matrix) computations related to the computation of $\mathrm{ext}^i_D(M,D)$ and $\mathrm{ext}^j_D(\mathrm{ext}^i_D(M,D),D)$.

Similarly, if N a finitely generated right D-module and $\mathcal{G}$ a right D-module, then using a free resolution (4) of N, we can define the following abelian groups:

$$\begin{cases} \mathrm{ext}^0_D(N,\mathcal{G}) = \hom_D(N,\mathcal{G}) \cong \ker_{\mathcal{G}}(.S_1), \\ \mathrm{ext}^i_D(N,\mathcal{G}) \cong \ker_{\mathcal{G}}(.S_{i+1})/\mathrm{im}_{\mathcal{G}}(.S_i), \quad i \geq 1. \end{cases}$$

We note that if M is a left (resp., right) D-module, then $\mathrm{ext}^i_D(M,D)$ is a right (resp., left) D-module due to the $D-D$-bimodule structure of D (see, e.g., [50]).

Definition 3 ([50])**.** A left D-module $\mathcal{F}$ is *injective* if $\mathrm{ext}^i_D(M,\mathcal{F}) = 0$ for all left D-modules M and for all $i \geq 1$.

Example 2. If Ω is an open convex subset of $\mathbb{R}^n$, then the space $C^\infty(\Omega)$ (resp., $\mathcal{D}'(\Omega)$, $\mathcal{S}'(\Omega)$, $\mathcal{A}(\Omega)$, $\mathcal{O}(\Omega)$) of smooth functions (resp., distributions/temperate distributions, real analytic/ holomorphic functions) on Ω is an injective $D = k[\partial_1,\ldots,\partial_n]$-module ($k = \mathbb{R}$, $\mathbb{C}$) [34, 36, 53].

If M is a finitely generated left D-module and $\mathcal{F}$ an injective left D-module, then applying the contravariant left exact functor $\hom_D(\,\cdot\,,\mathcal{F})$ to (3), and using Theorem 1 and the fact that $\mathrm{ext}^i_D(\,\cdot\,,\mathcal{F}) = 0$ for all $i \geq 1$, we obtain the following exact sequence of abelian groups:

$$\ldots \xleftarrow{R_3.} \mathcal{F}^{p_2} \xleftarrow{R_2.} \mathcal{F}^{p_1} \xleftarrow{R_1.} \mathcal{F}^{p_0} \longleftarrow \hom_D(M,\mathcal{F}) \longleftarrow 0.$$

The contravariant functor $\hom_D(\,\cdot\,,\mathcal{F})$ is then said to be *exact*. Since $\ker_{\mathcal{F}}(R_{i+1}.) = R_i\,\mathcal{F}^{p_{i-1}}$ for all $i \geq 1$, the linear system $\ker_{\mathcal{F}}(R_{i+1}.)$ is then *parametrized* by R_i (called a *parametrization*).

Let us now state two results which will be used in Section 3.

Theorem 3 ([50])**.** *Let (2) be a short exact sequence of left (resp., right) D-modules and N a left (resp., right) D-module. Then, the following long exact sequence holds*

$$\begin{array}{rlclcl} 0 & \longrightarrow \mathrm{ext}^0_D(M'',N) & \xrightarrow{g^\star} & \mathrm{ext}^0_D(M,N) & \xrightarrow{f^\star} & \mathrm{ext}^0_D(M',N) \\ & \longrightarrow \mathrm{ext}^1_D(M'',N) & \longrightarrow & \mathrm{ext}^1_D(M,N) & \longrightarrow & \mathrm{ext}^1_D(M',N) \\ & \longrightarrow \mathrm{ext}^2_D(M'',N) & \longrightarrow & \mathrm{ext}^2_D(M,N) & \longrightarrow & \ldots, \end{array}$$

where $f^\star$ (resp., $g^\star$) is defined by $f^\star(\phi) = \phi \circ f$ (resp., $g^\star(\psi) = \psi \circ g$) for all $\phi \in \hom_D(M,N)$ (resp., for all $\psi \in \hom_D(M'',N)$).

Remark 2. One can prove that a left D-module M is projective iff $\mathrm{ext}^i_D(M,N) = 0$ for all left D-module N and for all $i \geq 1$ (see, e.g., [50]). If P and P' are the two projective left D-modules considered in Remark 1, the *additivity* of the functor $\mathrm{ext}^i_D(\,\cdot\,,N)$ (see, e.g., [50]) then yields

$$\forall\, i \geq 1, \quad \begin{cases} \mathrm{ext}^i_D(\ker\pi \oplus P',N) \cong \mathrm{ext}^i_D(\ker\pi,N) \oplus \mathrm{ext}^i_D(P',N) = \mathrm{ext}^i_D(\ker\pi,N), \\ \mathrm{ext}^i_D(\ker\pi' \oplus P,N) \cong \mathrm{ext}^i_D(\ker\pi',N) \oplus \mathrm{ext}^i_D(P,N) = \mathrm{ext}^i_D(\ker\pi',N), \end{cases}$$

and thus, $\mathrm{ext}^i_D(\ker\pi,N) \cong \mathrm{ext}^i_D(\ker\pi',N)$ for $i \geq 1$, which shows that $\mathrm{ext}^i_D(\ker\pi,N)$ depends only on M and N (up to isomorphism) for $i \geq 1$.

Combining Remark 2 with Theorem 3, we obtain the following result.

Proposition 1 ([50])**.** *Let (2) be a short exact sequence of left (resp., right) D-modules and M a projective left (resp., right) D-module. Then, for every left (resp., right) D-module N, we have $\mathrm{ext}^{i+1}_D(M'',N) \cong \mathrm{ext}^i_D(M',N)$ for $i \geq 1$.*

Let us introduce important invariants of modules and rings.

Definition 4 ([50])**.**

1. The *left projective dimension* of a left D-module M, denoted by $\mathrm{lpd}_D(M)$, is the minimum of the lengths of projective resolutions of M. If no such integer exists, then we set $\mathrm{lpd}_D(M) = \infty$. Similarly for the *right projective dimension* $\mathrm{rpd}_D(N)$ of a right D-module N.
2. The *left global dimension* (resp., *right global dimension*) of a ring D, denoted by $\mathrm{lgd}(D)$ (resp., $\mathrm{rgd}(D)$), is the supremum of $\mathrm{lpd}_D(M)$ (resp., $\mathrm{rpd}_D(N)$) for all left D-modules M (resp., all right D-modules N).
3. If the left and the right global dimension of D coincide, then the common value is called the *global dimension* of D and is denoted by $\mathrm{gld}(D)$.

Proposition 2 ([10]). *Let D be a noetherian ring and M a finitely generated left D-module. Then, we have:*

$$\mathrm{lpd}_D(M) = \sup\{i \in \mathbb{N} \mid \mathrm{ext}^i_D(M, D) \neq 0\}.$$

Similarly for the right projective dimension $\mathrm{rpd}_D(N)$ *of a right D-module N.*

Proposition 3 ([50]). $\mathrm{lgd}(D) \leq n$ *iff* $\mathrm{ext}^i_D(M, N) = 0$ *for all left D-modules M and N, and for all $i > n$.*

Theorem 4 ([50]). *If D is a noetherian ring, then* $\mathrm{lgld}(D) = \mathrm{rgld}(D)$.

Example 3. If k is a field, then $\mathrm{gld}(k[x_1, \ldots, x_n]) = n$ [50]. If k is a field of characteristic 0, $k' = \mathbb{R}$ or $\mathbb{C}$, and $D = A_n(k)$, $B_n(k)$, $\hat{\mathcal{D}}_n(k)$, or $\mathcal{D}_n(k')$, then $\mathrm{gld}(D) = n$ [9, 10, 30].

We are now in a position to recall how the properties stated in Definition 1 can be checked by means of homological techniques for a *noetherian regular domain D*, namely a noetherian domain D of finite global dimension $\mathrm{gld}(D)$.

Theorem 5 ([1, 14, 30, 38, 40]). *Let D be a noetherian domain with a finite global dimension* $\mathrm{gld}(D) = n$, $M = D^{1\times p}/(D^{1\times q}\, R)$ *a finitely presented left D-module, and* $N = D^q/(R\, D^p)$ *the so-called* Auslander transpose *right D-module of M.*

1. *The following left D-isomorphism holds:*

$$t(M) \cong \mathrm{ext}^1_D(N, D). \tag{9}$$

2. *M is torsion-free iff* $\mathrm{ext}^1_D(N, D) = 0$.

3. *The following long exact sequence holds*

$$0 \longrightarrow \mathrm{ext}^1_D(N, D) \longrightarrow M \xrightarrow{\varepsilon} \hom_D(\hom_D(M, D), D) \longrightarrow \mathrm{ext}^2_D(N, D) \longrightarrow 0, \tag{10}$$

where ε is defined in 3 of Definition 1.

4. *M is reflexive iff* $\mathrm{ext}^i_D(N, D) = 0$ *for* $i = 1, 2$.

5. *M is projective iff* $\mathrm{ext}^i_D(N, D) = 0$ *for* $i = 1, \ldots, n$.

Remark 3. The Auslander transpose right D-module $N = D^q/(R\, D^p)$ depends on the left D-module $M = D^{1\times p}/(D^{1\times q}\, R)$ up to a projective equivalence: if $M \cong M' = D^{1\times p'}/(D^{1\times q'}\, R')$, then $N \oplus D^{(p+q')} \cong N' \oplus D^{(p'+q)}$, where $N' = D^{q'}/(R'\, D^{p'})$ [1]. See [18] for a constructive proof. Using Remark 2, the additivity of the functor $\mathrm{ext}^i_D(\,\cdot\,, \mathcal{F})$ (see, e.g., [50]) then yields $\mathrm{ext}^i_D(N, \mathcal{F}) \cong \mathrm{ext}^i_D(N', \mathcal{F})$ for all left D-modules $\mathcal{F}$ and for $i \geq 1$. Therefore, the results stated in Theorem 5 do not depend on the chosen presentation of M.

Theorem 5 was implemented in the OreModules package [15] for the class of Ore algebras of functional operators implemented in the Maple package `Ore_algebra` (e.g., PD, shift, difference, time-delay operators) for which Buchberger's algorithm terminates for any admissible term order and which computes a Gröbner basis [14]. Using the OreModules package, we can effectively check whether or not the left D-module $M = D^{1\times p}/(D^{1\times q}\, R)$ admits torsion elements or is torsion-free, reflexive or projective. For applications of Theorem 5 to mathematical systems theory and mathematical physics, see [15].

Let us recall how to compute the torsion left D-submodule $t(M)$ of $M = D^{1\times p}/(D^{1\times q}\, R)$. We first consider $Q \in D^{p\times m}$ such that $\ker_D(R.) = Q\, D^m$. Then, we get the exact sequence $0 \longleftarrow N \longleftarrow D^q \xleftarrow{R.} D^p \xleftarrow{Q.} D^m$. Then, 1 of Theorem 5 shows that the defect of exactness at $D^{1\times p}$ of the complex $D^{1\times q} \xrightarrow{.R} D^{1\times p} \xrightarrow{.Q} D^{1\times m}$ is defined by

$$\mathrm{ext}^1_D(N, D) \cong t(M) = \ker_D(.Q)/\mathrm{im}_D(.R) = (D^{1\times q'}\, R')/(D^{1\times q}\, R), \tag{11}$$

where $R' \in D^{q'\times p}$ is any matrix such that $\ker_D(.Q) = D^{1\times q'}\, R'$. Moreover, the standard *third isomorphism theorem* [50] then yields:

$$M/t(M) = [D^{1\times p}/(D^{1\times q}\, R)]/[(D^{1\times q'}\, R')/(D^{1\times q}\, R)] \cong D^{1\times p}/(D^{1\times q'}\, R'). \tag{12}$$

We note that a right analogous of Theorem 1 asserts that $\hom_D(M, D) \cong \ker_D(R.)$. Hence, if $\hom_D(M, D) = 0$, then $0 \longleftarrow N \longleftarrow D^q \xleftarrow{R.} D^p \longleftarrow 0$ is an exact sequence, and thus the defect of exactness of the complex $D^{1\times q} \xrightarrow{.R} D^{1\times p} \longrightarrow 0$ at $D^{1\times p}$ is $\mathrm{ext}^1_D(N, D) \cong t(M) = D^{1\times p}/(D^{1\times q}\, R) = M$ by (9), i.e., M is a torsion left D-module. Conversely, if M is a torsion left D-module and $f \in \hom_D(M, D)$, then for every $m \in M$, there exists $d \in D \setminus \{0\}$ such that $d\, m = 0$, which yields $d\, f(m) = f(d\, m) = 0$, and thus $f(m) = 0$ since D is a domain and $f(m) \in D$. Thus, $f = 0$, i.e., $\hom_D(M, D) = 0$. We obtain the following corollary of Theorem 5.

Corollary 1 (see, e.g., [14]). *Let M be a finitely generated left module over a noetherian domain D. Then, M is a torsion left D-module iff* $\hom_D(M, D) = 0$.

Let us now introduce a lemma which gives a finite presentation of a factor module.

Proposition 4 (see, e.g. [16]). *Let $R \in D^{q\times p}$ and $R' \in D^{q'\times p}$ satisfy $D^{1\times q}\, R \subseteq D^{1\times q'}\, R'$, i.e., are such that $R = R''\, R'$ for a certain $R'' \in D^{q\times q'}$. Moreover, let $R'_2 \in D^{r'\times q'}$ be a matrix such that* $\ker_D(.R') = D^{1\times r'}\, R'_2$*, and let π and π' be respectively the following canonical projections:*

$$\pi\colon D^{1\times q'}\, R' \longrightarrow (D^{1\times q'}\, R')/(D^{1\times q}\, R), \quad \pi'\colon D^{1\times q'} \longrightarrow D^{1\times q'}/(D^{1\times q}\, R'' + D^{1\times r'}\, R'_2).$$

Then, the left D-homomorphism ι defined by

$$\begin{array}{rcl} D^{1\times q'}/(D^{1\times q}\, R'' + D^{1\times r'}\, R'_2) & \xrightarrow{\iota} & (D^{1\times q'}\, R')/(D^{1\times q}\, R) \\ \pi'(\lambda) & \longmapsto & \pi(\lambda\, R'), \end{array} \tag{13}$$

is an isomorphism and its inverse ι^{-1} is defined by:

$$\begin{array}{rcl} (D^{1\times q'}\, R')/(D^{1\times q}\, R) & \xrightarrow{\iota^{-1}} & D^{1\times q'}/(D^{1\times q}\, R'' + D^{1\times r'}\, R'_2) \\ \pi(\lambda\, R') & \longmapsto & \pi'(\lambda). \end{array}$$

Applying Proposition 4 to the left D-module $t(M) = (D^{1\times q'}\, R')/(D^{1\times q}\, R)$, we obtain

$$t(M) \cong D^{1\times q'}/(D^{1\times q}\, R'' + D^{1\times r'}\, R'_2) = D^{1\times q'}/(D^{1\times (q+r')}\, (R''^T \quad R'^T_2)^T), \tag{14}$$

where $R'' \in D^{q\times q'}$ and $R'_2 \in D^{r'\times q'}$ are defined by $R = R''\, R'$ and $\ker_D(.R') = D^{1\times r'}\, R'_2$.

If $t(M) = 0$, then using (11), the complex $D^{1\times q} \xrightarrow{.R} D^{1\times p} \xrightarrow{.Q} D^{1\times m}$ is exact at $D^{1\times p}$, and thus it defines the beginning of a free resolution of the left D-module $L = D^{1\times m}/(D^{1\times q}\, Q)$. Up to isomorphism, a finitely generated torsion-free left D-module M can then be embedded into a finite free left D-module since $M = D^{1\times p}/(D^{1\times q}\, R) \cong \mathrm{im}_D(.Q) \subseteq D^{1\times m}$. If $\mathcal{F}$ is an injective left D-module, then applying the exact functor $\hom_D(\cdot\,,\mathcal{F})$ to the above beginning of a free resolution of L, we obtain the exact sequence $\mathcal{F}^q \xleftarrow{R.} \mathcal{F}^p \xleftarrow{Q.} \mathcal{F}^m$, i.e., $\ker_{\mathcal{F}}(R.) = Q\,\mathcal{F}^m$, i.e., Q is a parametrization of $\ker_{\mathcal{F}}(R.)$. The computation of parametrizations is implemented in the OreModules package. This package allows one to explicitly parametrize underdetermined linear functional systems appearing in mathematical physics and in control theory (see [15]).

The above techniques will be generalized in Section 3 to determine the so-called *grade filtration* of a finitely generated left D-module M.

To finish with this section, we shortly recall a few classical results on homomorphisms of finitely presented modules that will be used in the next sections.

Proposition 5 ([16, 18])**.** *Let $M = D^{1\times p}/(D^{1\times q}\,R)$ (resp., $M' = D^{1\times p'}/(D^{1\times q'}\,R')$) be a left D-module finitely presented by $R \in D^{q\times p}$ (resp., $R' \in D^{q'\times p'}$), and $\pi\colon D^{1\times p} \longrightarrow M$ (resp., $\pi'\colon D^{1\times p'} \longrightarrow M'$) the canonical projection onto M (resp., M'). Then, every $f \in \hom_D(M, M')$ is defined by $f(\pi(\lambda)) = \pi'(\lambda\,P)$ for all $\lambda \in D^{1\times p}$, where $P \in D^{p\times p'}$ satisfies $R\,P = Q\,R'$ for a certain $Q \in D^{q\times q'}$. Moreover, we have:*

1. $\ker f = (D^{1\times r}\,S)/(D^{1\times q}\,R)$, *where the matrix $S \in D^{r\times p}$ is defined by:*
$$\ker_D(.(P^T \quad R'^T)^T) = D^{1\times r}\,(S \quad -T), \quad T \in D^{r\times q'}.$$
In particular, f is injective iff there exists a matrix $F \in D^{r\times q}$ such that $S = F\,R$.
2. $\operatorname{im} f = (D^{1\times p}\,P + D^{1\times q'}\,R')/(D^{1\times q'}\,R') \cong \operatorname{coim} f = D^{1\times p}/(D^{1\times r}\,S)$.
3. $\operatorname{coker} f = D^{1\times p'}/(D^{1\times p}\,P + D^{1\times q'}\,R')$. *Thus, f is surjective iff $(P^T \quad R'^T)^T$ admits a left inverse over D, i.e., $X \in D^{p'\times p}$ and $Y \in D^{p'\times q'}$ exist such that $X\,P + Y\,R' = I_{p'}$.*
4. *f is an isomorphism, i.e., $M \cong M'$, iff there exists $F \in D^{r\times q}$ such that $S = F\,R$ and the matrix $(P^T \quad R'^T)^T$ admits a left inverse over D. If $X \in D^{p'\times p}$ is defined as in 3, then $f^{-1} \in \hom_D(M', M)$ is defined by $f^{-1}(\pi'(\lambda')) = \pi(\lambda'\,X)$ for all $\lambda' \in D^{1\times p'}$.*

2.2 Baer's extensions

In this section, we give another interpretation of the abelian group $\operatorname{ext}^1_D(M, N)$ which will be used in Section 4. To do that, let us introduce a few more definitions (see, e.g., [50]).

Definition 5. 1. Let M and N be two left D-modules. An *extension of M by N* is a short exact sequence of left D-modules of the form:
$$e\colon\ 0 \longrightarrow N \xrightarrow{\alpha} E \xrightarrow{\beta} M \longrightarrow 0. \tag{15}$$

2. Two extensions $e_i\colon\ 0 \longrightarrow N \xrightarrow{\alpha_i} E_i \xrightarrow{\beta_i} M \longrightarrow 0$ of M by N for $i = 1, 2$ are said to be *equivalent*, which is denoted by $e_1 \sim e_2$, if there exists a left D-isomorphism $\phi\colon E_1 \longrightarrow E_2$ such that $\alpha_2 = \phi\circ\alpha_1$ and $\beta_1 = \beta_2\circ\phi$, or equivalently, such that the following commutative exact diagram holds:
$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r, "\alpha_1"] \arrow[d, equal] & E_1 \arrow[r, "\beta_1"] \arrow[d, "\phi"] & M \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & N \arrow[r, "\alpha_2"] & E_2 \arrow[r, "\beta_2"] & M \arrow[r] & 0. \end{tikzcd}$$

3. Let $[e]$ be the equivalence class of the extension e for the equivalence relation $\sim$. The set of all equivalence classes of extensions of M by N is denoted by $\mathrm{e}_D(M,N)$.

The next theorem, which can be traced back to Baer's work, plays an important role in homological algebra. In particular, it explains the terminology *extension* used for $\mathrm{ext}^1_D(M,N)$.

Theorem 6 ([50]). *Let M and N be two left D-modules. Then, we have:*

$$\mathrm{ext}^1_D(M,N) \cong \mathrm{e}_D(M,N).$$

The next theorem gives an explicit description of the isomorphism stated in Theorem 6 in the case where M and N are two finitely presented left D-modules.

Theorem 7 ([46, 47]). *Let $M = D^{1\times p}/(D^{1\times q}\,R)$ and $N = D^{1\times s}/(D^{1\times t}\,S)$, $\pi\colon D^{1\times p} \longrightarrow M$ (resp., $\delta\colon D^{1\times s} \longrightarrow N$) be the canonical projection onto M (resp., N), and $R_2 \in D^{r\times q}$ a matrix such that $\ker_D(.R) = D^{1\times r}\,R_2$, and $\Omega = \{X \in D^{q\times s} \mid \exists\, Y \in D^{r\times t}\colon R_2\,X = Y\,S\}$. Then, every equivalence class of extensions of M by N is defined by the following short exact sequence*

$$e\colon\; 0 \longrightarrow N \xrightarrow{\alpha} E \xrightarrow{\beta} M \longrightarrow 0, \tag{16}$$

where $E = D^{1\times(p+s)}/(D^{1\times(q+t)}\,L)$ and $L = \begin{pmatrix} R & -A \\ 0 & S \end{pmatrix} \in D^{(q+t)\times(p+s)}$ for a certain $A \in \Omega$,

$$\begin{array}{rcl} N & \xrightarrow{\alpha} & E \\ \delta(\mu) & \longmapsto & \varrho(\mu\,(0 \quad I_s)), \end{array} \qquad \begin{array}{rcl} E & \xrightarrow{\beta} & M \\ \varrho(\nu) & \longmapsto & \pi(\nu\,(I_p \quad 0)^T), \end{array}$$

and $\varrho\colon D^{1\times(p+s)} \longrightarrow E$ is the canonical projection onto E. Finally, the equivalence class $[e]$ depends only on the residue class $\epsilon(A)$ of the matrix A in the following abelian group:

$$\Omega/(R\,D^{p\times s} + D^{q\times t}\,S) \cong \mathrm{ext}^1_D(M,N). \tag{17}$$

Remark 4. The extension e of Theorem 7 is *trivial*, i.e., $E \cong N \oplus M$, iff there exist $U \in D^{p\times s}$ and $V \in D^{q\times t}$ such that $A = R\,U + V\,S$, i.e., iff $\epsilon(A) = 0$. If D is a commutative polynomial ring over a computable field k, then using Kronecker product and Gröbner/Janet bases, we can check whether or not this identity holds and if so, compute solutions U and V. See, e.g., [47, 54].

The next corollary shows how to determine $\epsilon(A)$ for a given extension e of M by N.

Corollary 2 ([47]). *With the notations of Theorem 7, let $e'\colon\; 0 \longrightarrow N \xrightarrow{u} F \xrightarrow{v} M \longrightarrow 0$ be an extension of the left D-module $M = D^{1\times p}/(D^{1\times q}\,R)$ by the left D-module $N = D^{1\times s}/(D^{1\times t}\,S)$, $\{f_j\}_{j=1,\ldots,p}$ (resp., $\{e_i\}_{i=1,\ldots,q}$) the standard basis of $D^{1\times p}$ (resp., $D^{1\times q}$), $y_j = \pi(f_j)$, and $z_j \in F$ a pre-image of y_j under v for all $j = 1,\ldots,p$. Then, we have $\sum_{j=1}^p R_{ij}\,z_j \in \operatorname{im} u$ for all $i = 1,\ldots,q$, and, since u is injective, there exists a unique $n_i \in N$ satisfying $u(n_i) = \sum_{j=1}^p R_{ij}\,z_j$. If we consider a pre-image $a_i \in D^{1\times s}$ of n_i under δ, i.e., $n_i = \delta(a_i)$ for all $i = 1,\ldots,q$, then the extensions e' and (16) are equivalent, where $E = D^{1\times(p+s)}/(D^{1\times(q+t)}\,L)$ and:*

$$L = \begin{pmatrix} R & -A \\ 0 & S \end{pmatrix} \in D^{(q+t)\times(p+s)}, \quad A = \begin{pmatrix} a_1 \\ \vdots \\ a_q \end{pmatrix} \in D^{q\times s}.$$

Equivalently, the following commutative exact diagram holds

$$\begin{array}{ccccccccc} & & D^{1\times q} & \xrightarrow{.R} & D^{1\times p} & \xrightarrow{\pi} & M & \longrightarrow & 0 \\ & & \downarrow \phi & & \downarrow \psi & & \| & & \\ 0 & \longrightarrow & N & \xrightarrow{u} & F & \xrightarrow{v} & M & \longrightarrow & 0, \end{array}$$

where ψ and ϕ are respectively defined by:

$$\begin{array}{rcl} \psi\colon D^{1\times p} & \longrightarrow & F \\ f_j & \longmapsto & z_j, \; j=1,\ldots,p, \end{array} \qquad \begin{array}{rcl} \phi\colon D^{1\times q} & \longrightarrow & N \\ e_i & \longmapsto & n_i=\delta(a_i), \; i=1,\ldots,q. \end{array}$$

Theorem 7 and Corollary 2 will be abundantly used in Section 4. For more results on Baer's extensions, examples and applications to mathematical systems theory, see [4, 46, 47, 50, 54].

The next proposition shows how the presentation of the left D-module E defining the extension of M by N (see Theorem 7) changes with the presentations of M and N.

Proposition 6. *With the notations of Theorem 7, let $M = D^{1\times p}/(D^{1\times q}\,R)$, $N = D^{1\times s}/(D^{1\times t}\,S)$, and $E = D^{1\times (p+s)}/(D^{1\times (q+t)}\,L)$ be three left D-modules defining the extension e of M by N (16). Moreover, let f and g be two left D-isomorphisms defined by*

$$\begin{array}{rcl} f\colon M = D^{1\times p}/(D^{1\times q}\,R) & \longrightarrow & M' = D^{1\times p'}/(D^{1\times q'}\,R') \\ \pi(\lambda) & \longmapsto & \pi'(\lambda\,P), \end{array}$$

$$\begin{array}{rcl} g\colon N = D^{1\times s}/(D^{1\times t}\,S) & \longrightarrow & N' = D^{1\times s'}/(D^{1\times t'}\,S') \\ \delta(\mu) & \longmapsto & \delta'(\mu\,X), \end{array}$$

where π' (resp., δ') is the canonical projection onto M' (resp., N'), i.e., $P \in D^{p\times p'}$, $X \in D^{s\times s'}$ are such that there exist $Q \in D^{q\times q'}$, $P' \in D^{p'\times p}$, $Q' \in D^{q'\times q}$, $Y \in D^{t\times t'}$, $X' \in D^{s'\times s}$, $Y' \in D^{t'\times t}$, $T \in D^{p\times q}$, $T' \in D^{p'\times q'}$, $Z \in D^{s\times t}$, and $Z' \in D^{s'\times t'}$ satisfying the following identities:

$$\left\{\begin{array}{l} R\,P = Q\,R', \\ R'\,P' = Q'\,R, \\ I_p = P\,P' + T\,R, \\ I_{p'} = P'\,P + T'\,R', \end{array}\right. \qquad \left\{\begin{array}{l} S\,X = Y\,S', \\ S'\,X' = Y'\,S, \\ I_s = X\,X' + Z\,S, \\ I_{s'} = X'\,X + Z'\,S'. \end{array}\right. \tag{18}$$

Then, the extension e yields the following extension of M' by N'

$$e'\colon\; 0 \longrightarrow N' \xrightarrow{\alpha\circ g^{-1}} E \xrightarrow{f\circ\beta} M' \longrightarrow 0, \tag{19}$$

which implies that the left D-module E admits the following presentation

$$L' = \begin{pmatrix} R' & -Q'\,A\,X \\ 0 & S' \end{pmatrix} \in D^{(q'+t')\times(p'+s')},$$

i.e., $E \cong E' = D^{1\times(p'+s')}/(D^{1\times(q'+t')}\,L')$, where this left D-isomorphism is explicitly defined by

$$\begin{array}{rcl} \varphi\colon E & \longrightarrow & E' \\ \varrho(\nu) & \longmapsto & \varrho'(\nu\,U), \end{array} \qquad \begin{array}{rcl} \varphi^{-1}\colon E' & \longrightarrow & E \\ \varrho'(\nu') & \longmapsto & \varrho(\nu'\,U'), \end{array}$$

$$U = \begin{pmatrix} P & T\,A\,X \\ 0 & X \end{pmatrix} \in D^{(p+s)\times(p'+s')}, \quad U' = \begin{pmatrix} P' & 0 \\ 0 & X' \end{pmatrix} \in D^{(p'+s')\times(p+s)},$$

and $\varrho'\colon D^{1\times(p'+s')} \longrightarrow E'$ is the canonical projection onto E'.

Proof. With the notations (18), 4 of Proposition 5 yields:

$$\begin{array}{rcl} f^{-1}\colon M' = D^{1\times p'}/(D^{1\times q'}\,R') & \longrightarrow & M = D^{1\times p}/(D^{1\times q}\,R) \\ \pi'(\lambda') & \longmapsto & \pi(\lambda'\,P'), \\ g^{-1}\colon N' = D^{1\times s'}/(D^{1\times t'}\,S') & \longrightarrow & N = D^{1\times s}/(D^{1\times t}\,S) \\ \delta'(\mu') & \longmapsto & \delta(\mu'\,X'). \end{array}$$

Using (18), we get $(I_q - Q\,Q' - R\,T)\,R = R - Q\,Q'\,R - R\,T\,R = R - R\,P\,P' - R\,T\,R = 0$. Thus, if $\ker_D(.R) = D^{1\times r}\,R_2$, then there exists $T_2 \in D^{q\times r}$ such that:

$$I_q = Q\,Q' + R\,T + T_2\,R_2. \tag{20}$$

Now, clearly, (16) yields (19). Moreover, since $A \in \Omega$ (see Theorem 7), there exists $B \in D^{r\times s}$ such that $R_2\,A = B\,S$. Hence, using this identity, (18) and (20), we obtain

$$\begin{aligned} L\,U &= \begin{pmatrix} R & -A \\ 0 & S \end{pmatrix} \begin{pmatrix} P & T\,A\,X \\ 0 & X \end{pmatrix} = \begin{pmatrix} R\,P & (R\,T - I_q)\,A\,X \\ 0 & S\,X \end{pmatrix} \\ &= \begin{pmatrix} Q\,R' & -(Q\,Q'\,A + T_2\,(R_2\,A))\,X \\ 0 & Y\,S' \end{pmatrix} = \begin{pmatrix} Q\,R' & -(Q\,Q'\,A + T_2\,B\,S)\,X \\ 0 & Y\,S' \end{pmatrix} \\ &= \begin{pmatrix} Q\,R' & -Q\,Q'\,A\,X - T_2\,B\,Y\,S' \\ 0 & Y\,S' \end{pmatrix} = \begin{pmatrix} Q & -T_2\,B\,Y \\ 0 & Y \end{pmatrix} \begin{pmatrix} R' & -Q'\,A\,X \\ 0 & S' \end{pmatrix} = V\,L', \end{aligned}$$

where V is the first matrix appearing in the last but one equality, which shows that φ is well-defined by Proposition 5. Similarly, using (18), we get

$$\begin{aligned} L'\,U' &= \begin{pmatrix} R' & -Q'\,A\,X \\ 0 & S' \end{pmatrix} \begin{pmatrix} P' & 0 \\ 0 & X' \end{pmatrix} = \begin{pmatrix} R'\,P' & -Q'\,A\,X\,X' \\ 0 & S'\,X' \end{pmatrix} \\ &= \begin{pmatrix} Q'\,R & -Q'\,A\,(I_s - Z\,S) \\ 0 & Y'\,S \end{pmatrix} = \begin{pmatrix} Q' & Q'\,A\,Z \\ 0 & Y' \end{pmatrix} \begin{pmatrix} R & -A \\ 0 & S \end{pmatrix} = V'\,L, \end{aligned}$$

where V' is the first matrix appearing in the last but one equality, which yields $\phi \in \hom_D(E', E)$ defined by $\phi(\varrho'(\nu')) = \varrho(\nu'\,U')$ for all $\nu' \in D^{1\times(p'+s')}$ by Proposition 5. Using (18), we also have

$$\begin{aligned} U\,U' &= \begin{pmatrix} P & T\,A\,X \\ 0 & X \end{pmatrix} \begin{pmatrix} P' & 0 \\ 0 & X' \end{pmatrix} = \begin{pmatrix} P\,P' & T\,A\,X\,X' \\ 0 & X\,X' \end{pmatrix} \\ &= \begin{pmatrix} I_p - T\,R & T\,A\,(I_s - Z\,S) \\ 0 & I_s - Z\,S \end{pmatrix} = I_{p+s} - \begin{pmatrix} T & -T\,A\,Z \\ 0 & Z \end{pmatrix} \begin{pmatrix} R & -A \\ 0 & S \end{pmatrix}, \end{aligned}$$

which shows that $\phi \circ \varphi = \mathrm{id}_E$. Moreover, using (18), we obtain

$$(P'\,T - T'\,Q')\,R = P'\,T\,R - T'\,Q'\,R = P'\,T\,R - T'\,R'\,P' = P'\,(I_p - P\,P') - (I_{p'} - P'\,P)\,P' = 0,$$

which shows that there exists $L \in D^{p'\times r}$ such that $P'\,T - T'\,Q' = L\,R_2$. Using $R_2\,A = B\,S$ and $S\,X = Y\,S'$ (see (18)), $(P'\,T - T'\,Q')\,A\,X = L\,(R_2\,A)\,X = L\,B\,S\,X = L\,B\,Y\,S'$, and then

$$\begin{aligned} U\,U' &= \begin{pmatrix} P' & 0 \\ 0 & X' \end{pmatrix} \begin{pmatrix} P & T\,A\,X \\ 0 & X \end{pmatrix} = \begin{pmatrix} P'\,P & P'\,T\,A\,X \\ 0 & X'\,X \end{pmatrix} \\ &= \begin{pmatrix} I_{p'} - T'\,R' & P'\,T\,A\,X \\ 0 & I_{s'} - Z'\,S' \end{pmatrix} = I_{p'+s'} - \begin{pmatrix} T' & -L\,B\,Y \\ 0 & Z' \end{pmatrix} \begin{pmatrix} R' & -Q'\,A\,X \\ 0 & S' \end{pmatrix}, \end{aligned}$$

which shows that $\varphi \circ \phi = \mathrm{id}_{E'}$, and thus proves that φ is a left D-isomorphism and $\phi = \varphi^{-1}$. $\square$

2.3 Pure modules and grade filtration

Let us introduce the concept of the *grade number* of a finitely generated left D-module M.

Definition 6 ([9, 10])**.** The *grade number* of a nonzero finitely generated left D-module M is defined by $j_D(M) = \inf\{i \in \mathbb{N} \mid \mathrm{ext}^i_D(M, D) \neq 0\}$. If $M = 0$, then we set $j_D(M) = \infty$. A similar definition holds for right D-modules.

If $M \neq 0$, then $j_D(M)$ is then the smallest integer such that $\mathrm{ext}^{j_D(M)}_D(M, D) \neq 0$.

Remark 5. If $\mathrm{gld}(D)$ is finite and M is a nonzero left D-module, then using Proposition 3, $\mathrm{ext}^i_D(M, D) = 0$ for all $i > \mathrm{gld}(D)$, which yield $0 \leq j_D(M) \leq \mathrm{gld}(D)$.

Let us now introduce the concept of *pure module* that will play an important role.

Definition 7 ([10])**.** A finitely generated left D-module M is said to be *pure* or $j_D(M)$*-pure* if $j_D(N) = j_D(M)$ for all nonzero left D-submodules N of M.

Remark 6. If M is a pure left D-module, then for every $m \in M \setminus \{0\}$, the cyclic left D-module $D\,m$ generated by m satisfies $j_D(D\,m) = j_D(M)$. More generally, if N is a left D-submodule of a $j_D(M)$-pure left D-module M, then N is also a $j_D(M)$-pure left D-module since every left D-submodule of N is a left D-submodule of M and $j_D(N) = j_D(M)$.

In what follows, we shall mainly focus on the class of *Auslander regular rings*.

Definition 8 ([10])**.** A ring D is called an *Auslander regular ring* if D is a noetherian ring of finite global dimension $\mathrm{gld}(D)$ which satisfies the *Auslander condition*, namely, for every $i \in \mathbb{N}$, for every finitely generated left (resp., right) D-module M, and for every left (resp., right) D-submodule N of $\mathrm{ext}^i_D(M, D)$, then $j_D(N) \geq i$.

Remark 7. If D is an Auslander regular ring, then for a nonzero finitely generated left D-module M, taking $N = \mathrm{ext}^i_D(M, D)$ in Definition 8, we get $j_D(\mathrm{ext}^i_D(M, D)) \geq i$, i.e., $\mathrm{ext}^j_D(\mathrm{ext}^i_D(M, D), D) = 0$ for $0 \leq j < i$. Similarly, considering $\mathrm{ext}^i_D(M, D)$ instead of M in Definition 8, then $N \subseteq \mathrm{ext}^i_D(\mathrm{ext}^i_D(M, D), D) \neq 0$ yields $j_D(N) \geq i$.

Theorem 8 ([10])**.** *Let D be an Auslander regular ring and M a nonzero finitely generated left D-module. Then, we have:*

1. *M is pure iff M is a left D-submodule of* $\mathrm{ext}^{j_D(M)}_D(\mathrm{ext}^{j_D(M)}_D(M, D), D)$.
2. *M is pure iff* $\mathrm{ext}^i_D(\mathrm{ext}^i_D(M, D), D) = 0$ *for* $i \neq j_D(M)$.
3. *If* $\mathrm{ext}^i_D(\mathrm{ext}^i_D(M, D), D) \neq 0$*, then* $\mathrm{ext}^i_D(\mathrm{ext}^i_D(M, D), D)$ *is a pure left D-module with grade number i, i.e.,* $j_D(\mathrm{ext}^i_D(\mathrm{ext}^i_D(M, D), D)) = i$.

Example 4. By 1 of Theorem 8, M is 0-pure iff M is a left D-submodule of $\hom_D(\hom_D(M, D), D)$. If D is a domain, then using 3 of Theorem 5, we deduce that M is 0-pure iff M is a torsion-free left D-module. In particular, the left D-module $M/t(M)$ is either zero or 0-pure.

Let us now show that pure modules naturally appear in the study of a finitely generated left module M over an Auslander regular ring D. Let us consider:

$$t_i(M) = \{m \in M \mid j_D(D\,m) \geq i\}, \quad i = 0, \ldots, n = \mathrm{gld}(D), \quad t_{n+1}(M) = 0. \tag{21}$$

To prove that the $t_i(M)$'s are left D-modules, we need the following important result.

Proposition 7 ([10])**.** *If $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ is a short exact sequence of left modules over an Auslander regular ring D, then:*

$$j_D(M) = \inf\{j_D(M'), j_D(M'')\}.$$

Remark 8. If $\mathrm{ext}^i_D(M', D) = 0$ and $\mathrm{ext}^i_D(M'', D) = 0$ for $0 \leq i \leq j$, then Theorem 3 yields $\mathrm{ext}^i_D(M, D) = 0$ for $0 \leq i \leq j$, which shows that $j_D(M) \geq \inf\{j_D(M'), j_D(M'')\}$. Thus, the Auslander regularity condition is only used to prove the other inequality.

Let us now explain why $t_i(M)$ is a left D-module. If $m \in t_i(M)$ and $d \in D$, then $d\,m \in D\,m$, i.e., $D\,(d\,m) \subseteq D\,m$. Then, applying Proposition 7 to the following short exact sequence

$$0 \longrightarrow D\,(d\,m) \longrightarrow D\,m \longrightarrow D\,m/D\,(d\,m) \longrightarrow 0,$$

we get $j_D(D\,(d\,m)) \geq j_D(D\,m) \geq i$, i.e., $d\,m \in t_i(M)$. Let us now consider m_1 and $m_2 \in t_i(M)$. Then, we have $m_1 + m_2 \in D\,m_1 + D\,m_2$. Since $D\,(m_1 + m_2) \subseteq D\,m_1 + D\,m_2$, similarly as previously, Proposition 7 yields $j_D(D\,(m_1 + m_2)) \geq j_D(D\,m_1 + D\,m_2)$. Now, applying again Proposition 7 to the following two standard short exact sequences

$$\begin{gathered} 0 \longrightarrow D\,m_1 \cap D\,m_2 \longrightarrow D\,m_1 \oplus D\,m_2 \longrightarrow D\,m_1 + D\,m_2 \longrightarrow 0, \\ 0 \longrightarrow D\,m_1 \longrightarrow D\,m_1 \oplus D\,m_2 \longrightarrow D\,m_2 \longrightarrow 0, \end{gathered}$$

(see, e.g., [50]), we then obtain the following inequality and equality

$$\left\{ \begin{array}{l} j_D(D\,m_1 + D\,m_2) \geq j_D(D\,m_1 \oplus D\,m_2), \\ j_D(D\,m_1 \oplus D\,m_2) = \inf\{j_D(D\,m_1), j_D(D\,m_2)\} = i, \end{array} \right.$$

which yields $j_D(D\,(m_1 + m_2)) \geq i$, i.e., $m_1 + m_2 \in t_i(M)$.

If M' is a left D-submodule of M such that $j_D(M') \geq i$ and if $m' \in M' \setminus \{0\}$, then applying Proposition 7 to the short exact sequence $0 \longrightarrow D\,m' \longrightarrow M' \longrightarrow M'/(D\,m') \longrightarrow 0$, we get $j_D(D\,m') \geq j_D(M') \geq i$, i.e., $m' \in t_i(M)$, and thus $M' \subseteq t_i(M)$, which proves that $t_i(M)$ is the largest left D-submodule L of M (D is a noetherian ring) which satisfies $j_D(L) \geq i$.

Note that $t_0(M) = \{m \in M \mid j_D(D\,m) \geq 0\} = M$. Thus, the following filtration of M holds:

$$0 = t_{n+1}(M) \subseteq t_n(M) \subseteq t_{n-1}(M) \subseteq \cdots \subseteq t_1(M) \subseteq t_0(M) = M. \tag{22}$$

If D is a domain, then using Corollary 1, we get $t_1(M) = t(M)$ since:

$$m \in t(M) \;\Leftrightarrow\; \mathrm{ext}^0_D(D\,m, D) = 0 \;\Leftrightarrow\; j_D(D\,m) \geq 1 \;\Leftrightarrow\; m \in t_1(M).$$

It can easily been seen that a module M is i-pure iff $t_i(M) = M$ and $t_{i+1}(M) = 0$.

Lemma 1. *The left D-module $t_i(M)/t_{i+1}(M)$ is either zero or is i-pure.*

Proof. Let us suppose that $P = t_i(M)/t_{i+1}(M)$ is nonzero. Applying Proposition 7 to the short exact sequence $0 \longrightarrow t_{i+1}(M) \longrightarrow t_i(M) \longrightarrow P \longrightarrow 0$, we get $j_D(P) \geq j_D(t_i(M)) \geq i$, and thus $P \subseteq t_i(P) \subseteq P$, i.e., $t_i(P) = P$. Let us now check that $t_{i+1}(P) = 0$, which will prove the result. Composing the two canonical projections $\alpha \colon t_i(M) \longrightarrow P = t_i(M)/t_{i+1}(M)$ and $\beta \colon P \longrightarrow P/t_{i+1}(P)$, we get the following commutative exact diagram:

$$
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & t_{i+1}(M) \arrow[d] & \ker(\beta \circ \alpha) \arrow[d] & \\
 & 0 \arrow[r] & t_i(M) \arrow[r, equal] \arrow[d, "\alpha"] & t_i(M) \arrow[r] \arrow[d, "\beta \circ \alpha"] & 0 \\
0 \arrow[r] & t_{i+1}(P) \arrow[r] & P \arrow[r, "\beta"] \arrow[d] & P/t_{i+1}(P) \arrow[r] \arrow[d] & 0. \\
 & & 0 & 0 &
\end{tikzcd}
$$

The *snake lemma* (see, e.g., [50]) then yields the following short exact sequence:

$$0 \longrightarrow t_{i+1}(M) \longrightarrow \ker(\beta \circ \alpha) \longrightarrow t_{i+1}(P) \longrightarrow 0.$$

Using Proposition 7, we have $j_D(\ker(\beta \circ \alpha)) = \inf\{j_D(t_{i+1}(M)), j_D(t_{i+1}(P))\} \geq i + 1$. Since $t_{i+1}(M) \subseteq \ker(\beta \circ \alpha) \subseteq t_i(M) \subseteq M$, we obtain $\ker(\beta \circ \alpha) = t_{i+1}(M)$, and thus $t_{i+1}(P) = 0$ by the above short exact sequence. □

According to Lemma 1, (22) is called the *grade filtration* (*purity filtration*) of M (see [10]).

Theorem 9 ([9, 10, 11])**.** *Let D be a ring equipped with a filtration $\{D_r\}_{r \geq -1}$ ($D_{-1} = 0$) such that the associated graded ring $\mathrm{gr}(D) = \bigoplus_{r \in \mathbb{N}} D_r/D_{r-1}$ satisfies the following three properties:*

1. $\mathrm{gr}(D)$ *is a commutative ring.*

2. $\mathrm{gr}(D)$ *is a noetherian ring.*

3. $\mathrm{gr}(D)$ *is a* regular ring of pure dimension $d \in \mathbb{N}$, *namely,* $\mathrm{gld}(\mathrm{gr}(D)_{\mathfrak{m}})$ *is equal to d for all localizations* $\mathrm{gr}(D)_{\mathfrak{m}}$ *of* $\mathrm{gr}(D)$ *at maximal ideals* $\mathfrak{m}$ *of* $\mathrm{gr}(D)$*.*

Then, the following results hold:

1. $\mathrm{gld}(\mathrm{gr}(D)_{\mathfrak{m}})$ *is equal to the* Krull dimension $\mathrm{Kdim}(\mathrm{gr}(D)_{\mathfrak{m}})$ *of the noetherian local ring* $\mathrm{gr}(D)_{\mathfrak{m}}$*, which also equal to the dimension* $\dim_{\mathrm{gr}(D)_{\mathfrak{m}}/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2)$ *of* $\mathfrak{m}/\mathfrak{m}^2$ *as a* $\mathrm{gr}(D)_{\mathfrak{m}}/\mathfrak{m}$*-vector space. This common value d for all maximal ideals $\mathfrak{m}$ of* $\mathrm{gr}(D)$ *is denoted by* $\dim(D)$*.*

2. *If $M \neq 0$ is a left D-module M, then the* characteristic ideal $J(M)$ *of* $\mathrm{gr}(D)$*, defined by*

$$J(M) = \sqrt{\mathrm{ann}_{\mathrm{gr}(D)}(\mathrm{gr}(M))} = \{a \in \mathrm{gr}(D) \mid \exists\, k \in \mathbb{N} \colon a^k \, \mathrm{gr}(M) = 0\},$$

does not depend on any good filtration of M (e.g., if $M = \sum_{i=j}^{p} D\, y_j$ then $\{M_r\}_{r \in \mathbb{N}}$ defined by $M_r = \sum_{j=1}^{p} D_r\, y_j$ for all $r \in \mathbb{N}$ is a good filtration of M, and $\mathrm{gr}(M) = \sum_{j=1}^{p} \mathrm{gr}(D)\, y_j$).

3. *If the* dimension *of M is defined by* $\dim_D(M) = \mathrm{Kdim}(\mathrm{gr}(D)/J(M))$*, then*

$$j_D(M) = \dim(D) - \dim_D(M), \tag{23}$$

i.e., the codimension *of M is equal to the grade number of M.*

A ring D satisfying (23) is called a *Cohen-Macaulay ring*. A natural substitute for $\dim_D(\cdot)$ for more general k-algebras is the so-called *Gel'fand-Kirillov dimension* GKdim (see, e.g., [35]).

If D satisfies the hypotheses of Theorem 9, then $\dim(D) = \mathrm{gld}(\mathrm{gr}(D))$ since we have $\mathrm{gld}(\mathrm{gr}(D)) = \sup_{\mathfrak{m}\in\mathrm{Max}(\mathrm{gr}(D))} \mathrm{gld}(\mathrm{gr}(D)_{\mathfrak{m}})$, where $\mathrm{Max}(\mathrm{gr}(D))$ is the set of the maximal ideals of $\mathrm{gr}(D)$ (see, e.g., [50]).

Example 5. If k is a field of characteristic 0 and A a *differential field* (namely, a field with a differential ring structure) of characteristic 0 (e.g., k, $k(x_1, \ldots, x_n)$), or $k[x_1, \ldots, x_n]$, $k[\![x_1, \ldots, x_n]\!]$, $k'\{x_1, \ldots, x_n\}$ where $k' = \mathbb{R}$ or $\mathbb{C}$, then the ring $D = A\langle \partial_1, \ldots, \partial_n\rangle$ of PD operators with coefficients in A is Auslander regular and Cohen-Macaulay (see [9, 10, 11]). In particular, if $\{D_i\}_{i\geq -1}$ is the *order filtration* of D, namely D_i is the A-submodule of D formed by the PD operators of order less than or equal to i, and χ_i is the class of ∂_i in D_1/D_0, then $\mathrm{gr}(D) = A[\chi_1, \ldots, \chi_n]$. Thus, if A is a differential field of characteristic 0 (e.g., k, $k(x_1, \ldots, x_n)$), then $\dim(D) = n$, and if $A = k[x_1, \ldots, x_n]$, $k[\![x_1, \ldots, x_n]\!]$, or $k'\{x_1, \ldots, x_n\}$, then $\dim(A) = n$ and $\dim(D) = 2\,n$.

Corollary 3 ([9, 10, 11])**.** *Let D be an Auslander regular ring and a Cohen-Macaulay ring, and M a nonzero finitely generated left D-module. Then, we have:*

1. $\dim_D(\mathrm{ext}^i_D(M, D)) \leq \dim(D) - i$.

2. $\dim_D(\mathrm{ext}^{j_D(M)}_D(M, D)) = \dim(D) - j_D(M)$.

3. *If* $\mathrm{ext}^i_D(\mathrm{ext}^i_D(M, D), D) \neq 0$*, then* $\dim_D(\mathrm{ext}^i_D(\mathrm{ext}^i_D(M, D), D)) = \dim(D) - i$.

4. *If M is an i-pure left D-module, then* $\dim_D(M) = \dim(D) - i$.

If D is an Auslander regular ring with $\mathrm{gld}(D) = n$, then a nonzero finitely generated left D-module M is called *holonomic* (resp., *subholonomic*) if $j_D(M) = n$ (resp., $j_D(M) \geq n-1$). It is convenient to assume that $M = 0$ is also holonomic so that M is holonomic if $j_D(M) \geq n$. If D is also a Cohen-Macaulay ring, then $M \neq 0$ is holonomic (resp., subholonomic) iff $\dim_D(M) = \dim(D) - n$ (resp., $\dim_D(M) \leq \dim(D) - n + 1$). In particular, if D is one of the rings of PD operators defined in Example 5, then we find again the classical definitions of holonomic and subholonomic modules over a ring of PD operators (see, e.g., [9, 10, 11, 33]).

Let us state a few remarks on holonomic modules. If $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ is a short exact sequence and $j_D(M') = j_D(M'') = i$, then $j_D(M) = i$ by Proposition 7. In particular, if M' and M'' are two holonomic left D-modules, so is M. The converse result also holds since Proposition 7 and $j_D(M) \geq n$ yield $j_D(M') \geq n$ and $j_D(M'') \geq n$. Thus, M is a holonomic left D-module iff M' and M'' are two holonomic left D-modules. Finally, a *simple* module (i.e., a module containing no nontrivial submodules) left $A_n(k)$-module is not necessarily holonomic as shown in [52]. But, a simple module over an Auslander regular ring D is pure.

3 Grade filtration

The goal of the section is to show how the grade filtration (22) of a finitely generated left module M over an Auslander regular ring D can be explicitly computed. Since we are motivated by developing an effective algorithm which can be implemented in computer algebra systems, in what follows, we shall only use free resolutions of modules and not the more general projective resolutions. This extension can easily be done and it is left to the interested reader.

Let D be a noetherian regular ring, i.e., a noetherian domain D with a finite global dimension $\mathrm{gld}(D) = n$, and M a finitely generated left D-module. Let us consider a free resolution of M:

$$0 \longleftarrow M \xleftarrow{\pi} D^{1\times p_0} \xleftarrow{.R_1} D^{1\times p_1} \xleftarrow{.R_2} \ldots \xleftarrow{.R_{i-1}} D^{1\times p_{i-1}} \xleftarrow{.R_i} D^{1\times p_i} \xleftarrow{.R_{i+1}} \ldots \tag{24}$$

Using (7) and Proposition 3, the defects of exactness of the following complex

$$0 \longrightarrow D^{p_0} \xrightarrow{R_1.} D^{p_1} \xrightarrow{R_2.} \ldots \xrightarrow{R_{i-1}.} D^{p_{i-1}} \xrightarrow{R_i.} D^{p_i} \xrightarrow{R_{i+1}.} D^{p_{i+1}} \xrightarrow{R_{i+2}.} \ldots \tag{25}$$

are the right D-modules defined by:

$$\begin{cases} \mathrm{ext}^0_D(M,D) \cong \ker_D(R_1.), \\ \mathrm{ext}^i_D(M,D) \cong \ker_D(R_{i+1}.)/(R_i\, D^{p_{i-1}}), \quad 1 \le i \le n, \\ \mathrm{ext}^i_D(M,D) = 0, \quad i > n. \end{cases} \tag{26}$$

To characterize the $\mathrm{ext}^i_D(M,D)$'s for all $0 \le i \le n$, we need to study $\ker_D(R_{i+1}.)$. For $1 \le k \le n+1$, considering the beginning of a free resolution of the finitely generated right D-module $\ker_D(R_k.)$, we obtain the following long exact sequence of right D-modules

$$D^{p_{(-1)k}} \xrightarrow{R_{0k}.} D^{p_{0k}} \xrightarrow{R_{1k}.} D^{p_{1k}} \xrightarrow{R_{2k}.} \ldots \xrightarrow{R_{(k-1)k}.} D^{p_{(k-1)k}} \xrightarrow{R_{kk}.} D^{p_{kk}} \xrightarrow{\kappa_{kk}} N_{kk} \longrightarrow 0, \tag{27}$$

where for k from 1 to $n+1$, we have set $R_{kk} = R_k$, $p_{kk} = p_k$, $p_{(k-1)k} = p_{k-1} = p_{(k-1)(k-1)}$ and:

$$N_{kk} = \mathrm{coker}_D(R_{kk}.) = D^{p_{kk}}/(R_{kk}\, D^{p_{(k-1)k}}).$$

Let us explain why this choice of the notations is natural. If we consider a squared-line paper sheet where each square has coordinates $(j,k) \in \mathbb{N}^2$, and if the long exact sequence (27) is placed at k^{th} level with $D^{p_{jk}}$ at position (j,k), then the horizontal arrow of the right D-homomorphism $R_{jk}.$ arrives at $D^{p_{jk}}$ with $j \le k$ (a good mnemonic device). For instance, the first three horizontal exact sequences can be arranged as follows:

$$\begin{array}{ccccccccccccc} D^{p_{-13}} & \xrightarrow{R_{03}.} & D^{p_{03}} & \xrightarrow{R_{13}.} & D^{p_{13}} & \xrightarrow{R_{23}.} & D^{p_{23}} & \xrightarrow{R_{33}.} & D^{p_{33}} & \xrightarrow{\kappa_{33}} & N_{33} & \longrightarrow 0, \\ D^{p_{-12}} & \xrightarrow{R_{02}.} & D^{p_{02}} & \xrightarrow{R_{12}.} & D^{p_{12}} & \xrightarrow{R_{22}.} & D^{p_{22}} & \xrightarrow{\kappa_{22}} & N_{22} & \longrightarrow & 0, & \\ D^{p_{-11}} & \xrightarrow{R_{01}.} & D^{p_{01}} & \xrightarrow{R_{11}.} & D^{p_{11}} & \xrightarrow{\kappa_{11}} & N_{11} & \longrightarrow & 0. & & & \end{array}$$

Since (25) is a complex, $R_{kk}\, R_{(k-1)(k-1)} = R_k\, R_{k-1} = 0$ for all $k = 2, \ldots, n+1$, and thus $R_{(k-1)(k-1)}\, D^{p_{(k-2)(k-1)}} \subseteq \ker_D(R_{kk}.) = R_{(k-1)k}\, D^{p_{(k-2)k}}$, which shows the existence of a matrix $F_{(k-2)k} \in D^{p_{(k-2)k} \times p_{(k-2)(k-1)}}$ such that:

$$\forall\, k = 2, \ldots, n+1, \quad R_{(k-1)(k-1)} = R_{(k-1)k}\, F_{(k-2)k}. \tag{28}$$

Then, using (28), we get $R_{(k-1)k}\, F_{(k-2)k}\, R_{(k-2)(k-1)} = R_{(k-1)(k-1)}\, R_{(k-2)(k-1)} = 0$, i.e.,

$$F_{(k-2)k}\ R_{(k-2)(k-1)}\, D^{p_{(k-3)(k-1)}} \subseteq \ker_D(R_{(k-1)k}\,.) = R_{(k-2)k} D^{p_{(k-3)k}},$$

and thus, there exists a matrix $F_{(k-3)k} \in D^{p_{(k-3)k} \times p_{(k-3)(k-1)}}$ such that:

$$\forall\ k = 2, \ldots, n+1, \quad F_{(k-2)k}\ R_{(k-2)(k-1)} = R_{(k-2)k}\, F_{(k-3)k}. \tag{29}$$

Similarly, we can show that for $k = 3, \ldots, n+1$, there exist matrices $F_{(k-j)k} \in D^{p_{(k-j)k} \times p_{(k-j)(k-1)}}$ with $j = 3, \ldots, k$ such that:

$$F_{(k-j)k}\, R_{(k-j)(k-1)} = R_{(k-j)k}\, F_{(k-j-1)k}. \tag{30}$$

Let us denote by:

$$R_{00} = 0, \quad N_{00} = D^{p_{00}}/0 \cong D^{p_{00}}, \quad p_{01} = p_{00}, \quad p_{-10} = 0. \tag{31}$$

Using (27), (28), (29), (30) and (31), we get the following commutative diagram formed by $n+2$ horizontal exact sequences (where to reduce the size of the diagram, we set $m = n+1$):

$$\begin{tikzcd}
D^{p_{-1m}} \arrow[r, "R_{0m}\cdot"] & D^{p_{0m}} \arrow[r, "R_{1m}\cdot"] & D^{p_{1m}} \arrow[r, "R_{2m}\cdot"] & D^{p_{2m}} \arrow[r, "R_{3m}\cdot"] & D^{p_{3m}} \arrow[r, "R_{4m}\cdot"] & D^{p_{4m}} \arrow[r, "R_{5m}\cdot"] & D^{p_{5m}} \arrow[r, "R_{6m}\cdot"] & \ldots \\
D^{p_{-1n}} \arrow[u, "F_{-1m}\cdot"'] \arrow[r, "R_{0n}\cdot"] & D^{p_{0n}} \arrow[u, "F_{0m}\cdot"'] \arrow[r, "R_{1n}\cdot"] & D^{p_{1n}} \arrow[u, "F_{1m}\cdot"'] \arrow[r, "R_{2n}\cdot"] & D^{p_{2n}} \arrow[u, "F_{2m}\cdot"'] \arrow[r, "R_{3n}\cdot"] & D^{p_{3n}} \arrow[u, "F_{3m}\cdot"'] \arrow[r, "R_{4n}\cdot"] & D^{p_{4n}} \arrow[u, "F_{4m}\cdot"'] \arrow[r, "R_{5n}\cdot"] & D^{p_{5n}} \arrow[u, "F_{5m}\cdot"'] \arrow[r, "R_{6n}\cdot"] & \ldots \\
\vdots \arrow[u, "F_{-1n}\cdot"'] & \vdots \arrow[u, "F_{0n}\cdot"'] & \vdots \arrow[u, "F_{1n}\cdot"'] & \vdots \arrow[u, "F_{2n}\cdot"'] & \vdots \arrow[u, "F_{3n}\cdot"'] & \vdots \arrow[u, "F_{4n}\cdot"'] & \vdots \arrow[u, "F_{5n}\cdot"'] & \\
D^{p_{-13}} \arrow[u, "F_{-14}\cdot"'] \arrow[r, "R_{03}\cdot"] & D^{p_{03}} \arrow[u, "F_{04}\cdot"'] \arrow[r, "R_{13}\cdot"] & D^{p_{13}} \arrow[u, "F_{14}\cdot"'] \arrow[r, "R_{23}\cdot"] & D^{p_{23}} \arrow[u, "F_{24}\cdot"'] \arrow[r, "R_{33}\cdot"] & D^{p_{33}} \arrow[u, equal] \arrow[r, "\kappa_{33}"] & N_{33} \arrow[r] & 0 & \\
D^{p_{-12}} \arrow[u, "F_{-13}\cdot"'] \arrow[r, "R_{02}\cdot"] & D^{p_{02}} \arrow[u, "F_{03}\cdot"'] \arrow[r, "R_{12}\cdot"] & D^{p_{12}} \arrow[u, "F_{13}\cdot"'] \arrow[r, "R_{22}\cdot"] & D^{p_{22}} \arrow[u, equal] \arrow[r, "\kappa_{22}"] & N_{22} \arrow[r] & 0 & & \\
D^{p_{-11}} \arrow[u, "F_{-12}\cdot"'] \arrow[r, "R_{01}\cdot"] & D^{p_{01}} \arrow[u, "F_{02}\cdot"'] \arrow[r, "R_{11}\cdot"] & D^{p_{11}} \arrow[u, equal] \arrow[r, "\kappa_{11}"] & N_{11} \arrow[r] & 0 & & & \\
0 \arrow[r] & D^{p_{00}} \arrow[u, equal] \arrow[r, "\kappa_{00}"] & N_{00} \arrow[r] & 0. & & & &
\end{tikzcd} \tag{32}$$

Now, if we denote by $N_{(k-j)k}$ the finitely presented right D-module defined by

$$N_{(k-j)k} = \operatorname{coker}_D(R_{(k-j)k}\,.) = D^{p_{(k-j)k}}/(R_{(k-j)k}\, D^{p_{(k-j-1)k}}),$$

then, (32) can be truncated to get the following commutative diagram formed by horizontal exact sequences:

$$
\begin{tikzcd}
D^{p_{-1(n+1)}} \arrow[r, "R_{0(n+1)}\cdot"] & D^{p_{0(n+1)}} \arrow[r, "R_{1(n+1)}\cdot"] & D^{p_{1(n+1)}} \arrow[r, "\kappa_{1(n+1)}"] & N_{1(n+1)} \arrow[r] & 0 \\
D^{p_{-1n}} \arrow[u, "F_{-1(n+1)}\cdot"'] \arrow[r, "R_{0n}\cdot"] & D^{p_{0n}} \arrow[u, "F_{0(n+1)}\cdot"'] \arrow[r, "R_{1n}\cdot"] & D^{p_{1n}} \arrow[u, "F_{1(n+1)}\cdot"'] \arrow[r, "\kappa_{1n}"] & N_{1n} \arrow[r] & 0 \\
\vdots \arrow[u, "F_{-1n}\cdot"'] & \vdots \arrow[u, "F_{0n}\cdot"'] & \vdots \arrow[u, "F_{1n}\cdot"'] & \vdots & \\
D^{p_{-13}} \arrow[u, "F_{-14}\cdot"'] \arrow[r, "R_{03}\cdot"] & D^{p_{03}} \arrow[u, "F_{04}\cdot"'] \arrow[r, "R_{13}\cdot"] & D^{p_{13}} \arrow[u, "F_{14}\cdot"'] \arrow[r, "\kappa_{13}"] & N_{13} \arrow[r] & 0 \\
D^{p_{-12}} \arrow[u, "F_{-13}\cdot"'] \arrow[r, "R_{02}\cdot"] & D^{p_{02}} \arrow[u, "F_{03}\cdot"'] \arrow[r, "R_{12}\cdot"] & D^{p_{12}} \arrow[u, "F_{13}\cdot"'] \arrow[r, "\kappa_{12}"] & N_{12} \arrow[r] & 0 \\
D^{p_{-11}} \arrow[u, "F_{-12}\cdot"'] \arrow[r, "R_{01}\cdot"] & D^{p_{01}} \arrow[u, "F_{02}\cdot"'] \arrow[r, "R_{11}\cdot"] & D^{p_{11}} \arrow[u, equal] \arrow[r, "\kappa_{11}"] & N_{11} \arrow[r] & 0 \\
0 \arrow[r] & D^{p_{00}} \arrow[u, equal] \arrow[r, "\kappa_{00}"] & N_{00} \arrow[r] & 0. &
\end{tikzcd}
\tag{33}
$$

For $k = 1, \ldots, n+1$ and $j = 0, \ldots, k-1$, using the exactness of the following complex

$$D^{p_{(k-j-2)k}} \xrightarrow{R_{(k-j-1)k}\cdot} D^{p_{(k-j-1)k}} \xrightarrow{R_{(k-j)k}\cdot} D^{p_{(k-j)k}}$$

at $D^{p_{(k-j-1)k}}$, we get $N_{(k-j-1)k} = \operatorname{coker}_D(R_{(k-j-1)k}.) \cong \operatorname{im}_D(R_{(k-j)k}.)$ which, when combined with the short exact sequence $0 \longrightarrow \operatorname{im}_D(R_{(k-j)k}.) \longrightarrow D^{p_{(k-j)k}} \xrightarrow{\kappa_{(k-j)k}} N_{(k-j)k} \longrightarrow 0$, yields the following short exact sequence of right D-modules:

$$0 \longrightarrow N_{(k-j-1)k} \longrightarrow D^{p_{(k-j)k}} \longrightarrow N_{(k-j)k} \longrightarrow 0. \tag{34}$$

Using (26), we obtain the following characterization of the right D-modules $\operatorname{ext}^i_D(M, D)$'s:

$$\begin{cases} \operatorname{ext}^i_D(M, D) \cong \ker_D(R_{(i+1)(i+1)}.)/\operatorname{im}_D(R_{ii}.) = (R_{i(i+1)}\, D^{p_{(i-1)(i+1)}})/(R_{ii}\, D^{p_{(i-1)i}}), \\ \hfill 0 \leq i \leq n, \\ \operatorname{ext}^i_D(M, D) = 0, \quad i > n. \end{cases} \tag{35}$$

Since $N_{ii} = D^{p_{ii}}/(R_{ii}\, D^{p_{(i-1)i}})$, $N_{i(i+1)} = D^{p_{i(i+1)}}/(R_{i(i+1)}\, D^{p_{(i-1)(i+1)}})$, $p_{i(i+1)} = p_{ii}$, and $N_{00} = D^{p_{00}}$, (35) and the *third isomorphism theorem* of module theory (see, e.g., [50]) yield the following short exact sequence of right D-modules:

$$0 \longrightarrow \operatorname{ext}^i_D(M, D) \longrightarrow N_{ii} \longrightarrow N_{i(i+1)} \longrightarrow 0, \quad i = 0, \ldots, n. \tag{36}$$

Applying the contravariant left exact functor $\hom_D(\,\cdot\,,D)$ to the short exact sequence of (36) and using Theorem 3, we obtain the following long exact sequences:

$$\begin{array}{cccccccc} 0 & \longrightarrow & \mathrm{ext}^0_D(N_{01},D) & \longrightarrow & \mathrm{ext}^0_D(N_{00},D) & \longrightarrow & \mathrm{ext}^0_D(\mathrm{ext}^0_D(M,D),D) & \\ & \longrightarrow & \mathrm{ext}^1_D(N_{01},D) & \longrightarrow & \mathrm{ext}^1_D(N_{00},D), & & & \\ \cdots & \longrightarrow & \mathrm{ext}^{i-1}_D(N_{i(i+1)},D) & \longrightarrow & \mathrm{ext}^{i-1}_D(N_{ii},D) & \longrightarrow & \mathrm{ext}^{i-1}_D(\mathrm{ext}^i_D(M,D),D) & \\ & \longrightarrow & \mathrm{ext}^i_D(N_{i(i+1)},D) & \longrightarrow & \mathrm{ext}^i_D(N_{ii},D) & \longrightarrow & \mathrm{ext}^i_D(\mathrm{ext}^i_D(M,D),D) & \\ & \xrightarrow{\tau_{i+1}} & \mathrm{ext}^{i+1}_D(N_{i(i+1)},D) & \longrightarrow & \mathrm{ext}^{i+1}_D(N_{ii},D) & \longrightarrow & \ldots, & i=1,\ldots,n. \end{array} \tag{37}$$

In what follows, we shall assume that D satisfies the following property

$$\forall\, i \geq 1, \quad \mathrm{ext}^{i-1}_D(\mathrm{ext}^i_D(M,D),D)=0, \tag{38}$$

for all finitely generated left D-modules M. In particular, by Remark 7, this condition holds if D is an Auslander regular ring (see Definition 8).

We note that $\mathrm{ext}^1_D(N_{00},D)$ is reduced to 0 since $N_{00}=D^{p_{00}}$ is a free, and thus a projective right D-module (see Remark 2). Using (38), the above long exact sequences then yield the following long exact sequences of left D-modules:

$$\begin{array}{l} 0 \longrightarrow \mathrm{ext}^0_D(N_{01},D) \longrightarrow \mathrm{ext}^0_D(N_{00},D) \longrightarrow \mathrm{ext}^0_D(\mathrm{ext}^0_D(M,D),D) \longrightarrow \mathrm{ext}^1_D(N_{01},D) \longrightarrow 0, \\ 0 \longrightarrow \mathrm{ext}^i_D(N_{i(i+1)},D) \longrightarrow \mathrm{ext}^i_D(N_{ii},D) \longrightarrow \mathrm{ext}^i_D(\mathrm{ext}^i_D(M,D),D), \qquad i=1,\ldots,n. \end{array} \tag{39}$$

Applying Proposition 1 to (34) for $k=i+1$ and $j=0,\ldots,i-1$, i.e., to the short exact sequence $0 \longrightarrow N_{(i-j)(i+1)} \longrightarrow D^{p_{(i-j+1)(i+1)}} \longrightarrow N_{(i-j+1)(i+1)} \longrightarrow 0$, we obtain:

$$\forall\, i=1,\ldots,n, \quad \mathrm{ext}^{i+1}_D(N_{(i+1)(i+1)},D) \cong \mathrm{ext}^i_D(N_{i(i+1)},D) \cong \ \ldots\ \cong \mathrm{ext}^1_D(N_{1(i+1)},D). \tag{40}$$

Similarly, applying Proposition 1 to (34) for $k=i+1$ and $j=0$ gives:

$$\mathrm{ext}^{i+2}_D(N_{(i+1)(i+1)},D) \cong \mathrm{ext}^{i+1}_D(N_{i(i+1)},D). \tag{41}$$

Applying Proposition 1 to the above short exact sequence with $i=0$ and $j=0$, we get:

$$\mathrm{ext}^2_D(N_{11},D) \cong \mathrm{ext}^1_D(N_{01},D).$$

Thus, the first long exact sequence of (39) yields the following one

$$0 \longrightarrow \mathrm{ext}^0_D(N_{01},D) \xrightarrow{\gamma_{10}} \mathrm{ext}^0_D(N_{00},D) \xrightarrow{\gamma_{00}} \mathrm{ext}^0_D(\mathrm{ext}^0_D(M,D),D) \longrightarrow \mathrm{ext}^2_D(N_{11},D) \longrightarrow 0, \tag{42}$$

and (39) and (40) yield the following exact sequence of left D-modules

$$0 \longrightarrow \mathrm{ext}^{i+1}_D(N_{(i+1)(i+1)},D) \xrightarrow{\gamma_{(i+1)i}} \mathrm{ext}^i_D(N_{ii},D) \xrightarrow{\gamma_{ii}} \mathrm{ext}^i_D(\mathrm{ext}^i_D(M,D),D) \longrightarrow \mathrm{coker}\,\gamma_{ii} \longrightarrow 0, \tag{43}$$

where:

$$\forall\, i=1,\ldots,n, \quad \mathrm{coker}\,\gamma_{ii} \subseteq \mathrm{ext}^{i+1}_D(N_{i(i+1)},D) \cong \mathrm{ext}^{i+2}_D(N_{(i+1)(i+1)},D). \tag{44}$$

Hence, if we introduce the following finitely generated left D-modules

$$\forall\, i=0,\ldots,n+1, \quad T_i \triangleq \mathrm{ext}^i_D(N_{ii},D), \tag{45}$$

then (43) can be rewritten as the following exact sequences:

$$0 \longrightarrow T_{i+1} \xrightarrow{\gamma_{(i+1)i}} T_i \xrightarrow{\gamma_{ii}} \mathrm{ext}^i_D(\mathrm{ext}^i_D(M,D),D) \longrightarrow \mathrm{coker}\,\gamma_{ii} \longrightarrow 0, \quad i=1,\ldots,n. \tag{46}$$

Remark 9. If D is an Auslander regular ring, then using (45) and Remark 7, T_i is either zero or $j_D(T_i) \geq i$. Moreover, according to 3 of Theorem 8, $\mathrm{ext}_D^i(\mathrm{ext}_D^i(M, D), D)$ is either zero or is i-pure. In particular, $T_i/\gamma_{(i+1)i}(T_{i+1})$ is a left D-submodule of $\mathrm{ext}_D^i(\mathrm{ext}_D^i(M, D), D)$, and thus it is either zero or is i-pure by Remark 7. Finally, using Remark 7 and (44), we find that $\mathrm{coker}\, \gamma_{ii}$ is either zero or $j_D(\mathrm{coker}\, \gamma_{ii}) = j_D(\mathrm{ext}_D^{i+2}(N_{(i+1)(i+1)}, D)) \geq i + 2$.

Using (40), up to isomorphism, the left D-modules T_i's are the defects of exactness at $D^{1\times p_{0i}}$ of the horizontal complexes of the following commutative diagram (marked in red)

$$\begin{tikzcd}
D^{1\times p_{-1(n+1)}} \arrow[d, ".F_{-1(n+1)}"] & D^{1\times p_{0(n+1)}} \arrow[l, ".R_{0(n+1)}"'] \arrow[d, ".F_{0(n+1)}"] & D^{1\times p_{1(n+1)}} \arrow[l, ".R_{1(n+1)}"'] \arrow[d, ".F_{1(n+1)}"] \\
D^{1\times p_{-1n}} \arrow[d, ".F_{-1n}"] & D^{1\times p_{0n}} \arrow[l, ".R_{0n}"'] \arrow[d, ".F_{0n}"] & D^{1\times p_{1n}} \arrow[l, ".R_{1n}"'] \arrow[d, ".F_{1n}"] \\
\vdots \arrow[d, ".F_{-14}"] & \vdots \arrow[d, ".F_{04}"] & \vdots \arrow[d, ".F_{14}"] \\
D^{1\times p_{-13}} \arrow[d, ".F_{-13}"] & D^{1\times p_{03}} \arrow[l, ".R_{03}"'] \arrow[d, ".F_{03}"] & D^{1\times p_{13}} \arrow[l, ".R_{13}"'] \arrow[d, ".F_{13}"] \\
D^{1\times p_{-12}} \arrow[d, ".F_{-12}"] & D^{1\times p_{02}} \arrow[l, ".R_{02}"'] \arrow[d, ".F_{02}"] & D^{1\times p_{12}} \arrow[l, ".R_{12}"'] \arrow[d, equal] \\
D^{1\times p_{-11}} \arrow[d] & D^{1\times p_{01}} \arrow[l, ".R_{01}"'] \arrow[d, equal] & D^{1\times p_{11}} \arrow[l, ".R_{11}"'] \\
0 & D^{1\times p_{00}} \arrow[l] & 0, \arrow[l]
\end{tikzcd}$$

i.e., we have:

$$T_0 = D^{1\times p_{00}}, \quad T_i = \ker_D(.R_{0i})/\mathrm{im}_D(.R_{1i}), \quad i = 1, \ldots, n + 1. \tag{47}$$

If $\rho_i \colon \ker_D(.R_{0i}) \longrightarrow T_i = \ker_D(.R_{0i})/(D^{1\times p_{1i}}\, R_{1i})$ is the canonical projection onto the D-module T_i for $i = 1, \ldots, n + 1$, then $\gamma_{(i+1)i} \in \hom_D(T_{i+1}, T_i)$ (see (46)) is defined by:

$$\forall\, \lambda \in \ker_D(.R_{0(i+1)}), \quad \gamma_{(i+1)i}(\rho_{i+1}(\lambda)) = \rho_i(\lambda\, F_{0(i+1)}), \quad i = 1, \ldots, n. \tag{48}$$

The inclusion $\ker_D(.R_{01}) \subseteq D^{1\times p_{01}}$ yields the following commutative exact diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & D^{1\times p_{11}}\, R_{11} \arrow[r] \arrow[d, equal] & \ker_D(.R_{01}) \arrow[r, "\rho_1"] \arrow[d] & T_1 \arrow[r] \arrow[d, "\gamma_{10}"] & 0 \\
0 \arrow[r] & D^{1\times p_{11}}\, R_{11} \arrow[r] \arrow[d] & D^{1\times p_{01}} \arrow[r, "\pi"] & M \arrow[r] & 0, \\
 & 0 & & &
\end{tikzcd}$$

where $\gamma_{10} \in \hom_D(T_1, M)$ is defined by

$$\forall\, \lambda \in \ker_D(.R_{01}), \quad \gamma_{10}(\rho_1(\lambda)) = \pi(\lambda), \tag{49}$$

and π is the canonical projection onto $M = D^{1\times p_{01}}/(D^{1\times p_{11}}\, R_{11})$, i.e., $\gamma_{10} = \mathrm{id}_{T_1}$. In particular, γ_{10} is injective. Moreover, using $T_1 = \ker_D(.R_{01})/(D^{1\times p_{11}}\, R_{11}) \subseteq M = D^{1\times p_{01}}/(D^{1\times p_{11}}\, R_{11})$, the third isomorphism theorem of module theory (see, e.g., [50]) gives:

$$M/T_1 \cong D^{1\times p_{01}}/\ker_D(.R_{01}). \tag{50}$$

Finally, if D is a domain, then 1 of Theorem 5 shows that $T_1 = t(M)$ and $M/T_1 = M/t(M)$.

Let us now study the long exact sequences (42) and (46) for $i = n-1,\, n$.

A right D-module analogous of Theorem 1 shows that $\mathrm{ext}^0_D(N_{01}, D) \cong \ker_D(.R_{01})$. Using (31), $T_0 = \mathrm{ext}^0_D(N_{00}, D) = \hom_D(D^{p_{00}}, D) \cong D^{1\times p_{00}} = D^{1\times p_{01}}$ (see (47)). The long exact sequence (42) then becomes the following one:

$$0 \longrightarrow \ker_D(.R_{01}) \xrightarrow{\gamma_{10}} D^{1\times p_{01}} \xrightarrow{\gamma_{00}} \mathrm{ext}^0_D(\mathrm{ext}^0_D(M, D), D) \longrightarrow \mathrm{ext}^2_D(N_{11}, D) \longrightarrow 0.$$

Proposition 3, $\mathrm{gld}(D) = n$ and (44) yield $\operatorname{coker}\gamma_{(n-1)(n-1)} \subseteq \mathrm{ext}^{n+1}_D(N_{nn}, D) = 0$, i.e., $\operatorname{coker}\gamma_{(n-1)(n-1)} = 0$. Thus, setting $i = n-1$ in (46), we get the following short exact sequence

$$0 \longrightarrow T_n \xrightarrow{\gamma_{n(n-1)}} T_{n-1} \xrightarrow{\gamma_{(n-1)(n-1)}} \mathrm{ext}^{n-1}_D(\mathrm{ext}^{n-1}_D(M, D), D) \longrightarrow 0,$$

which shows that:

$$T_{n-1}/(\gamma_{n(n-1)}(T_n)) \cong \mathrm{ext}^{n-1}_D(\mathrm{ext}^{n-1}_D(M, D), D). \tag{51}$$

Proposition 3, $\mathrm{gld}(D) = n$ and (44) imply that $\operatorname{coker}\gamma_{nn} \subseteq \mathrm{ext}^{n+2}_D(N_{(n+1)(n+1)}, D) = 0$, i.e., $\operatorname{coker}\gamma_{nn} = 0$. By Proposition 3, we also have:

$$T_{n+1} = \mathrm{ext}^{n+1}_D(N_{(n+1)(n+1)}, D) = 0.$$

Thus, setting $i = n$ in (46), we obtain the following short exact sequence

$$0 \longrightarrow T_n \xrightarrow{\gamma_{nn}} \mathrm{ext}^n_D(\mathrm{ext}^n_D(M, D), D) \longrightarrow 0,$$

which shows that:

$$T_n \cong \mathrm{ext}^n_D(\mathrm{ext}^n_D(M, D), D). \tag{52}$$

Therefore, the following exact sequences of left D-modules hold

$$\begin{array}{ccccccccc}
 & & 0 & \longrightarrow & T_n & \xrightarrow{\gamma_{nn}} & \mathrm{ext}_D^n(\mathrm{ext}_D^n(M,D),D) & \longrightarrow 0, \\
0 \longrightarrow & & T_n & \xrightarrow{\gamma_{n(n-1)}} & T_{n-1} & \longrightarrow & \mathrm{coker}\,\gamma_{n(n-1)} & \longrightarrow 0, \\
 & & \vdots & & \vdots & & \vdots & \\
0 \longrightarrow & & T_i & \xrightarrow{\gamma_{i(i-1)}} & T_{i-1} & \longrightarrow & \mathrm{coker}\,\gamma_{i(i-1)} & \longrightarrow 0, \\
 & & \vdots & & \vdots & & \vdots & \\
0 \longrightarrow & & T_2 & \xrightarrow{\gamma_{21}} & T_1 & \longrightarrow & \mathrm{coker}\,\gamma_{21} & \longrightarrow 0, \\
0 \longrightarrow & & T_1 & \xrightarrow{\gamma_{10}} & M & \xrightarrow{\rho} & M/T_1 & \longrightarrow 0, \\
0 \longrightarrow & & \ker_D(.R_{01}) & \longrightarrow & D^{1\times p_{01}} & \xrightarrow{\pi'} & M/T_1 & \longrightarrow 0, \\
0 \longrightarrow & & M/T_1 & \longrightarrow & \mathrm{ext}_D^0(\mathrm{ext}_D^0(M,D),D) & \longrightarrow & \mathrm{ext}_D^2(N_{11},D) & \longrightarrow 0,
\end{array} \tag{53}$$

where:

$$\forall\, i=2,\ldots,n, \quad \mathrm{coker}\,\gamma_{i(i-1)} \subseteq \mathrm{ext}_D^i(\mathrm{ext}_D^i(M,D),D). \tag{54}$$

Now, since the $\gamma_{i(i-1)}$'s are injective left D-homomorphisms and $\gamma_{10} = \mathrm{id}_{T_1}$, we can define the following sequence $\{M_i\}_{i=0,\ldots,n}$ of left D-submodules of M as follows:

$$M_0 = M, \quad M_1 = \gamma_{10}(T_1) = T_1, \quad \forall\, i=2,\ldots,n, \quad M_i = (\gamma_{10}\circ\gamma_{21}\circ\gamma_{32}\circ\cdots\circ\gamma_{i(i-1)})(T_i) \cong T_i. \tag{55}$$

Using (48) and (49), the left D-module M_i can be explicitly characterized by:

$$\forall\, i=1,\ldots,n, \quad M_i = \pi(\ker_D(.R_{0i})\,(F_{0i}\,\ldots\,F_{02})). \tag{56}$$

The inclusion $\gamma_{i(i-1)}(T_i) \subseteq T_{i-1}$ yields $M_i \subseteq M_{i-1}$, and we get the following filtration of M:

$$0 = M_{n+1} \subseteq M_n \subseteq M_{n-1} \subseteq \cdots \subseteq M_2 \subseteq M_1 \subseteq M_0 = M. \tag{57}$$

Remark 10. Let us explain why the left D-modules M_i's depend only on M and not on the free resolution (24) of M. Using Remark 3, the Auslander transpose right D-module $N_{ii} = D^{p_{ii}}/(R_{ii}\,D^{p_{(i-1)i}})$ of the left D-module $\mathrm{coker}_D(.R_{ii}) = D^{1\times p_{ii}}/(D^{1\times p_{(i-1)i}}\,R_{ii})$ depends only on $\mathrm{coker}_D(.R_{ii})$ up to projective equivalence. Using Remark 1 and the exactness of the free resolution (24) of M, we find that the right D-modules

$$\begin{cases} \mathrm{coker}_D(.R_{ii}) = \mathrm{coker}_D(.R_i) \cong D^{1\times p_{i-1}}\,R_{i-1} = \ker_D(.R_{i-2}), \quad i \geq 3, \\ \mathrm{coker}_D(.R_{22}) = \mathrm{coker}_D(.R_2) = D^{1\times p_1}\,R_1 = \ker\pi, \\ \mathrm{coker}_D(.R_{11}) = \mathrm{coker}_D(.R_1) = M, \end{cases}$$

depend on M up to projective equivalence. Thus, the right D-module N_{ii} depends only on M up to a projective equivalence for $i \geq 1$. Using Remark 2, $M_i \cong T_i = \mathrm{ext}_D^i(N_{ii},D)$ finally depends only on M for $i \geq 1$ and not on the free resolution (24) of M.

Let us state a few consequences of the above results.

Corollary 4. *1. The following long exact sequences of left D-modules hold*

$$0 \longrightarrow M_{i+1} \xrightarrow{\iota_{i+1}} M_i \xrightarrow{\varepsilon_i} \mathrm{ext}_D^i(\mathrm{ext}_D^i(M,D),D) \longrightarrow C_i \longrightarrow 0, \quad i=0,\ldots,n, \tag{58}$$

where $C_i = \operatorname{coker} \varepsilon_i$ is isomorphic to a left D-submodule of $\operatorname{ext}_D^{i+2}(N_{(i+1)(i+1)}, D)$ for all $i = 0, \ldots, n-2$ (with equality for $i = 0$), $C_{n-1} = 0$, $C_n = 0$. In particular:

$$M_n \cong \operatorname{ext}_D^n(\operatorname{ext}_D^n(M, D), D), \quad M_{n-1}/M_n \cong \operatorname{ext}_D^{n-1}(\operatorname{ext}_D^{n-1}(M, D), D).$$

2. *If $M_i = 0$, then $M_i = M_{i+1} = \ldots = M_n = 0$.*

3. *$M = M_{j_D(M)}$.*

Proof. 1. Using the last short exact sequence of (53), $M = M_0$ and $M_1 = T_1$, we obtain (58) for $i = 0$, where $C_0 = \operatorname{ext}_D^2(N_{11}, D)$. Let us now suppose that $i = 1, \ldots, n$ and let $\alpha_i = \gamma_{10} \circ \gamma_{21} \circ \gamma_{32} \circ \cdots \circ \gamma_{i(i-1)}$ be the left D-isomorphism from T_i to M_i (see (55)). Then, the long exact sequence (46) yields (58) where $\iota_{i+1} = \alpha_i \circ \gamma_{(i+1)i} \circ \alpha_{i+1}^{-1} = \operatorname{id}_{M_{i+1}}$, $\varepsilon_i = \gamma_{ii} \circ \alpha_i^{-1}$ and $C_i = \operatorname{coker} \varepsilon_i \cong \operatorname{coker} \gamma_{ii} \subseteq \operatorname{ext}_D^{i+2}(N_{(i+1)(i+1)}, D)$ by (44). Since $\operatorname{gld}(D) = n$, we get $C_{n-1} = C_n = 0$. Finally, (58) for $i = n$, $M_{n+1} = 0$ and C_n yield $M_n \cong \operatorname{ext}_D^n(\operatorname{ext}_D^n(M, D), D)$, and (58) for $i = n-1$ and $C_{n-1} = 0$ implies that $M_{n-1}/M_n \cong \operatorname{ext}_D^{n-1}(\operatorname{ext}_D^{n-1}(M, D), D)$.

2. The equality is a direct consequence of (57).

3. If $j_D(M) = 0$, then the result holds since $M = M_0$. Let us suppose that $j_D(M) \geq 1$. Then, $\operatorname{ext}_D^i(\operatorname{ext}_D^i(M, D), D) = 0$ for $i = 0, \ldots, j_D(M)-1$ since $\operatorname{ext}_D^i(M, D) = 0$ for $i = 0, \ldots, j_D(M)-1$. Using (58), we get $M_{i+1} = M_i$ for $i = 1, \ldots, j_D(M) - 1$. Finally, the last short exact sequence of (53) yields $M/M_1 = 0$, i.e., $M = M_1$, which finally proves the result. $\square$

Let us give consequences of the above results for an Auslander regular ring D.

Proposition 8. *If D is an Auslander regular ring and $\operatorname{gld}(D) = n$, then we have:*

1. *If M_i is nonzero, then $j_D(M_i) \geq i$ for $i = 0, \ldots, n$.*

2. *If M_i/M_{i+1} is nonzero, then M_i/M_{i+1} is an i-pure left D-module for $i = 0, \ldots, n$. Moreover, if $M_{i+1} = 0$, then M_i is either zero or an i-pure left D-submodule of M. In particular, M_n is either zero or a n-pure left D-module.*

3. *If C_i is nonzero, then $j_D(C_i) \geq i + 2$ for $i = 0, \ldots, n-2$.*

4. *$M_i = M_{i+1}$ iff $\operatorname{ext}_D^i(\operatorname{ext}_D^i(M, D), D) = 0$.*

Proof. 1. Since $M_i \cong T_i = \operatorname{ext}_D^i(N_{ii}, D)$ for $i = 1, \ldots, n$, Remark 7 then shows that $j_D(M_i) \geq i$. Moreover, $M_0 = M$, and thus $j_D(M_0) \geq 0$.

2. By 3 of Theorem 8, $\operatorname{ext}_D^i(\operatorname{ext}_D^i(M, D), D)$ is either zero or i-pure, and so is the left D-module $M_i/M_{i+1} \cong \operatorname{im} \varepsilon_i \subseteq \operatorname{ext}_D^i(\operatorname{ext}_D^i(M, D), D)$ (see Remark 6). In particular, if $M_{i+1} = 0$, then M_i is either zero or an i-pure left D-submodule of M. Finally, $M_n \cong \operatorname{ext}_D^n(\operatorname{ext}_D^n(M, D), D)$ (see 1 of Corollary 4) implies that M_n is either zero or n-pure.

3. Since $C_i = \operatorname{coker} \varepsilon_i$ is isomorphic to a left D-submodule of $\operatorname{ext}_D^{i+2}(N_{(i+1)(i+1)}, D)$ for $i = 0, \ldots, n-2$ (see 1 of Corollary 4), then Remark 7 then yields $j_D(C_i) \geq i+2$ for $i = 0, \ldots, n-2$.

4. If $M_i = M_{i+1}$, then (58) gives $C_i \cong \operatorname{ext}_D^i(\operatorname{ext}_D^i(M, D), D)$. On the one hand, by 3 of Theorem 8, C_i is either zero or i-pure, and thus we either have $C_i = 0$ or $j_D(C_i) = i$. On the other hand, using 3, if $C_i \neq 0$, then $j_D(C_i) \geq i + 2$, which shows that $C_i = 0$. Conversely, if $\operatorname{ext}_D^i(\operatorname{ext}_D^i(M, D), D) = 0$, then (58) yields $M_i = M_{i+1}$. $\square$

If D is also a Cohen-Macaulay ring, then using Corollary 3, we obtain:

$$\forall\, i = 0, \ldots, n, \quad \dim_D(M_i) \leq \dim(D) - i, \quad \dim_D(M_i/M_{i+1}) = \dim(D) - i. \tag{59}$$

Let us now show that the filtration $\{M_i\}_{i=0,\ldots,n}$ of M defined by (55) is exactly the grade filtration $\{t_i(M)\}_{i=0,\ldots,n}$ of M defined in (21) when D is an Auslander regular ring.

Theorem 10. *Let D be an Auslander regular ring and M a finitely generated left D-module. Then, we have $t_i(M) = M_i$ for all $i = 0, \ldots, n = \mathrm{gld}(D)$, i.e., the grade filtration (22) of M and the filtration (7) of M coincide.*

Proof. Let us first prove that $0 \neq M_i \subseteq t_i(M)$. By 1 of Proposition 8, $j_D(M_i) \geq i$. If $m \in M_i$, then applying Proposition 7 to the short exact sequence $0 \longrightarrow D\,m \longrightarrow M_i \longrightarrow M_i/(D\,m) \longrightarrow 0$, we obtain $j_D(D\,m) \geq j_D(M_i) = i$, and thus $m \in t_i(M)$, i.e., $M_i \subseteq t_i(M)$.

Following [9], let us now prove $t_i(M) \subseteq M_i$ by induction on i, i.e., $t_i(M) = M_i$ by the above point. We first note that $t_0(M) = M = M_0$, which proves the result for $i = 0$. Let us now assume that $t_i(M) = M_i$ and let us show that it yields $t_{i+1}(M) = M_{i+1}$. Since $M_{i+1} \subseteq t_{i+1}(M) \subseteq t_i(M)$, we get $t_{i+1}(M)/M_{i+1} \subseteq t_i(M)/M_{i+1} = M_i/M_{i+1}$. Using 2 of Proposition 8, M_i/M_{i+1} is either zero or an i-pure left D-module. If $M_i/M_{i+1} = 0$, then $t_{i+1}(M)/M_{i+1} = 0$, i.e., $t_{i+1}(M) = M_{i+1}$, which proves the result. Hence, let us assume that M_i/M_{i+1} is an i-pure left D-module. Then, by definition of a pure module, its left D-submodule $t_{i+1}(M)/M_{i+1}$ is also either zero or i-pure. If $t_{i+1}(M)/M_{i+1}$ is i-pure, then $j_D(t_{i+1}(M)/M_{i+1}) = i$. But, applying Proposition 7 to the following short exact sequence

$$0 \longrightarrow M_{i+1} \longrightarrow t_{i+1}(M) \longrightarrow t_{i+1}(M)/M_{i+1} \longrightarrow 0$$

gives $j_D(t_{i+1}(M)/M_{i+1}) \geq j_D(t_{i+1}(M)) \geq i + 1$, which yields a contradiction. Thus, we obtain $t_{i+1}(M)/M_{i+1} = 0$, i.e., $t_{i+1}(M) = M_{i+1}$, which finally proves the result by induction. □

Remark 11. We can combine Theorem 10 and Proposition 8 to find again 2 of Theorem 8. Indeed, using Theorem 10, $M \neq 0$ is i-pure iff $M = M_1 = \ldots = M_i \neq 0$ and $M_{i+1} = M_{i+2} = \ldots = M_{n+1} = 0$. By 4 of Proposition 8, the equalities are equivalent to $\mathrm{ext}_D^k(\mathrm{ext}_D^k(M, D), D) = 0$ for $k = 0, \ldots, i-1$ and $k = i+1, \ldots, n$. Let us study the inequality. Combining $M_i \neq 0$, $M_{i+1} = 0$ and (58), $\mathrm{ext}_D^i(\mathrm{ext}_D^i(M, D), D)$ then contains the nonzero left D-submodule M_i, which shows that $\mathrm{ext}_D^i(\mathrm{ext}_D^i(M, D), D) \neq 0$. Since $\mathrm{ext}_D^i(\mathrm{ext}_D^i(M, D), D) \neq 0$ yields $M \neq 0$, $M \neq 0$ is then an i-pure left D-module iff $\mathrm{ext}_D^k(\mathrm{ext}_D^k(M, D), D) = 0$ for $k \neq i$ and $\mathrm{ext}_D^i(\mathrm{ext}_D^i(M, D), D) \neq 0$.

The existence of the filtration (57) only requires that D is a noetherian regular domain which satisfies (38). If D is an Auslander regular ring, then Theorem 10 proves that (57) is exactly the grade filtration (22) of M. If D is also a Cohen-Macaulay ring, then using (59), the filtration $\{M_i\}_{i=0,\ldots,n}$ of M gives a built-in classification of the elements of M by means of their (co)dimensions. This filtration is sometimes called the *codimension filtration* of M (or *equidimensional decomposition* in algebraic geometry).

Remark 12. If D satisfies the hypotheses of Theorem 9, then Theorem 9 shows that the *characteristic ideal* $J(M)$ of $\mathrm{gr}(D)$ does not depend on the choice of a good filtration of M. The *characteristic variety* of M is then defined by $\mathrm{char}(M) = \{\mathfrak{p} \in \mathrm{Spec}(\mathrm{gr}(D)) \mid J(M) \subseteq \mathfrak{p}\}$, where $\mathrm{Spec}(\mathrm{gr}(D))$ is the set of prime ideals of $\mathrm{gr}(D)$ endowed with the Zariski topology. A well-known result in algebraic analysis states that a short exact sequence of left D-modules

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

yields the equality $\mathrm{char}(M) = \mathrm{char}(M') \cup \mathrm{char}(M'')$ (see, e.g., [30, 33]). Applying this result to the short exact sequences $0 \longrightarrow M_{i+1} \longrightarrow M_i \longrightarrow M_i/M_{i+1} \longrightarrow 0$ for $i = 0, \dots, n$, we get:

$$\mathrm{char}(M) = \bigcup_{i=0,\dots,n} \mathrm{char}(M_i/M_{i+1}). \tag{60}$$

It can be proved that the characteristic variety $\mathrm{char}(P)$ of an i-pure module P is *equidimensional* in the sense that every irreducible component of $\mathrm{char}(P)$ has dimension $\dim(D) - i$ (see, e.g., [10]). Hence, (60) is an equidimensional decomposition of the affine algebraic variety $\mathrm{char}(M)$.

Theorem 10 shows that the grade filtration of M can be computed by means of elementary methods of module theory and homological algebra. In particular, we do not need to compute a *Cartan-Eilenberg resolution* $P^{\bullet\bullet}$ (see, e.g., [50]) of the complex (25) (called $\mathrm{Rhom}(M, D)$ within derived categories (see, e.g., [25])), the *total complex* $\mathrm{Tot}(\hom_D(P^{\bullet\bullet}, D))$ of the double complex $\hom_D(P^{\bullet\bullet}, D)$, and the spectral sequence associated with the first filtration of $\mathrm{Tot}(\hom_D(P^{\bullet\bullet}, D))$. For more details, see [2, 9, 10, 11, 23, 25, 32, 50]. Our approach has then the advantage to be easily implementable in any computer algebra system containing an implementation of Gröbner bases for (noncommutative) polynomial rings (e.g., Maple, Singular, Macaulay2, Magma, Mathematica). Another advantage will be explained in Section 4.

The filtration (57) is a particular case of the more general *bidualizing filtration* $\{M_i\}_{i=0,\dots,n}$ of a finitely generated module M over a regular ring D [9, 10], of which the existence can be proved by means of a spectral sequence argument. In this case, M_i/M_{i+1} is then a left D-*subquotient* (i.e., a quotient of a left D-submodule) of $\mathrm{ext}^i_D(\mathrm{ext}^i_D(M, D), D)$, and not simply a left D-submodule as shown above for an Auslander regular ring D. Finally, we note that the results developed in [9, 49] were extended in [32] for *an Auslander-Gorenstein ring* D, namely a noetherian ring of finite *injective dimension* m as a left/right D-module (i.e., $\mathrm{ext}^i_D(M, D) = 0$ for $i > m$ and for all finitely left/right D-modules M) [50] which satisfies the Auslander condition (see Definition 8).

Let us sum up the above results in the following algorithm.

Algorithm 1.
- **Input:** A noetherian regular ring D satisfying (38), $\mathrm{gld}(D) = n$, and $R \in D^{q \times p}$.
- **Output:** A sequence $\{T_i\}_{i=1,\dots,n}$ of finitely generated left D-modules defined by (45) and a sequence $\{\gamma_{10} \in \hom_D(T_1, M)\} \cup \{\gamma_{(i+1)i} \in \hom_D(T_{i+1}, T_i)\}_{i=1,\dots,n}$ of D-homomorphisms defined by (49) and (48) such that $\{M_i = (\gamma_{10} \circ \gamma_{21} \circ \gamma_{32} \circ \cdots \circ \gamma_{i(i-1)})(T_i)\}_{i=1,\dots,n}$ is a filtration of M (the grade filtration of M when D is an Auslander regular ring).

1. Set $R_1 = R$, $p_1 = p$, $p_2 = q$, and $M = D^{1 \times p_1}/(D^{1 \times p_2}\, R_1)$.
2. Compute matrices $R_k \in D^{p_k \times p_{k-1}}$ for $k = 2, \dots, n$ such that (24) is an exact sequence.
3. Set $p_{kk} = p_k$, $p_{(k-1)k} = p_{k-1} = p_{(k-1)(k-1)}$, $R_{kk} = R_k$, and $N_{kk} = D^{p_{kk}}/(R_{kk}\, D^{p_{(k-1)k}})$.
4. For $k = 1, \dots, n$ and for $j = 1, \dots, k$, compute matrices $R_{(k-j)k} \in D^{p_{(k-j)k} \times p_{(k-j-1)k}}$ such that (27) is an exact sequence.
5. For $k = 2, \dots, n$, compute matrices $F_{(k-2)k} \in D^{p_{(k-2)k} \times p_{(k-2)(k-1)}}$ such that:

$$R_{(k-1)(k-1)} = R_{(k-1)k}\, F_{(k-2)k}.$$

6. For $k = 2, \ldots, n$ and for $j = 2, \ldots, k$, compute $F_{(k-j)k} \in D^{p_{(k-j)k} \times p_{(k-j)(k-1)}}$ satisfying:

$$F_{(k-j)k}\, R_{(k-j)(k-1)} = R_{(k-j)k}\, F_{(k-j-1)k}.$$

7. Return the matrices R_{0i}, R_{1i}, and F_{0i} defining the left D-module $T_i = \ker_D(.R_{0i})/\mathrm{im}_D(.R_{1i})$ for $i = 1, \ldots, n$, $\gamma_{10} = \mathrm{id}_{T_1} \colon T_1 = \ker_D(.R_{01})/\mathrm{im}_D(.R_{11}) \longrightarrow M = D^{1\times p_{01}}/\mathrm{im}_D(.R_{11})$, and $\gamma_{i(i-1)} \in \hom_D(T_i, T_{i-1})$ by (48) for $i = 2, \ldots, n$.

Remark 13. Using 3 of Corollary 4, i.e., $M = M_{j_D(M)}$, let us explain how Algorithm 1 can then be speeded up when $j_D(M) \geq 1$ by avoiding the computation of the left D-modules T_i's for $i = 1, \ldots, j_D(M)$. Since $\mathrm{ext}^i_D(M, D) = 0$ for $i = 0, \ldots, j_D(M) - 1$, then (25) yields the following free resolution of $N_{j_D(M) j_D(M)}$:

$$D^{p_0} \xrightarrow{R_1.} D^{p_1} \xrightarrow{R_2.} \ldots \xrightarrow{R_{j_D(M)}.} D^{p_{j_D(M)}} \xrightarrow{\kappa_{j_D(M) j_D(M)}} N_{j_D(M) j_D(M)} \longrightarrow 0. \tag{61}$$

Applying Proposition 1 to (61), we get $\mathrm{ext}^{j_D(M)}_D(N_{j_D(M) j_D(M)}, D) \cong \mathrm{ext}^1_D(N_{11}, D) = M_1$, where $N_{11} = D^{p_1}/(R_1\, D^{p_0})$. Moreover, since $j_D(M) \geq 1$, $\hom_D(M, D) = 0$, and using Theorem 1, $\ker_D(R_1.) \cong \hom_D(M, D) = 0$, and thus $M_1 = \mathrm{ext}^1_D(N_{11}, D) \cong M$. Hence, we do not need to compute the beginning of a free resolution of the right D-module N_{kk} for $k = 1, \ldots, j_D(M)$, i.e., we can only consider $k = j_D(M) + 1, \ldots, n$ in 4 of Algorithm 1.

Algorithm 1 with its improvement explained in Remark 13 are implemented in the Maple package PurityFiltration [45] built upon OreModules [15]. The PurityFiltration package allows us to compute the grade filtration of a finitely generated left D-module M, where D is an Ore algebras available in OreModules. If an *involution* θ of D (namely, $\theta \colon D \longrightarrow D$ satisfies $\theta(d_1 + d_2) = \theta(d_1) + \theta(d_2)$, $\theta(d_1\, d_2) = \theta(d_2)\, \theta(d_1)$ for all d_1, $d_2 \in D$, and $\theta^2 = \mathrm{id}_D$) exists, then we can compute the matrices $R_{(k-j)k}$ defined in 4 of Algorithm 1 by left Gröbner basis techniques. For more details, see [14]. Algorithm 1 has also recently been implemented in the `homalg` based package `AbelianSystems` [7] by M. Barakat (University of Kaiserslautern) and the author.

Let us now determine a finite presentation of the left D-modules T_i's defined by (45). To do that, we first consider the beginning of a finite free resolution of $P_i = D^{1\times p_{-1i}}/(D^{1\times p_{0i}}\, R_{0i})$, namely, matrices $R'_{1i} \in D^{p'_{1i} \times p_{0i}}$ and $R'_{2i} \in D^{p'_{2i} \times p'_{1i}}$ such that $\ker_D(.R_{0i}) = D^{1\times p'_{1i}}\, R'_{1i}$ and $\ker_D(.R'_{1i}) = D^{1\times p'_{2i}}\, R'_{2i}$ for $i = 1, \ldots, n$. We obtain the commutative diagram (68) formed by horizontal exact sequences.

Remark 14. If $R_{0k} = 0$, i.e., $\ker_D(R_{1k}.) = 0$, then applying the functor $\hom_D(\,\cdot\,, D)$ to the short exact sequence $0 \longrightarrow D^{p_{0k}} \xrightarrow{R_{1k}.} D^{p_{1k}} \xrightarrow{\kappa_{1k}} N_{1k} \longrightarrow 0$, we get the following complex:

$$0 \longleftarrow D^{1\times p_{0k}} \xleftarrow{.R_{1k}} D^{1\times p_{1k}}.$$

Hence, we have $\ker_D(.R_{0k}) = D^{1\times p_{0k}}$, i.e., $R'_{1k} = I_{p_{0k}}$, $p'_{1k} = p_{0k}$, and $R'_{2k} = 0$.

Combining (56) with $\ker_D(.R_{0i}) = D^{1\times p'_{1i}}\, R'_{1i}$, we obtain the following explicit characterization of the M_i's, i.e., of the $t_i(M)$'s when D is an Auslander regular ring (see Theorem 10):

$$\begin{cases} M_1 = (D^{1\times p'_{11}}\, R'_{11})/(D^{1\times p_{11}}\, R_{11}), \\ M_i = (D^{1\times p'_{1i}}\, (R'_{1i}\, F_{0i}\, \ldots\, F_{02}))/(D^{1\times p_{11}}\, R_{11}), \quad i = 2, \ldots, n. \end{cases} \tag{62}$$

Hence, (62) shows that the residue classes of the rows of the matrix $R'_{1i}\, F_{0i}\, \ldots\, F_{02}$ in the left D-module $M = D^{1\times p_{01}}/(D^{1\times p_{11}}\, R_{11})$ generate the left D-module M_i.

Algorithm 2. • **Input:** A noetherian regular ring D satisfying (38), $\mathrm{gld}(D) = n$, and $R \in D^{q \times p}$.

- **Output:** A sequence $\{M_i\}_{i=1,\ldots,n}$ of left D-submodules of M defined by (62), i.e., the grade filtration (57) of M when D is an Auslander regular ring.

1. Apply Algorithm 1 to D and $R \in D^{q \times p}$ to obtain $R_{0i} \in D^{p_{0i} \times p_{-1i}}$ for $i = 1, \ldots, n$, and $F_{0i} \in D^{p_{0i} \times p_{0(i-1)}}$ for $i = 2, \ldots, n$.
2. Compute $R'_{1i} \in D^{p'_{1i} \times p_{0i}}$ such that $\ker_D(.R_{0i}) = D^{1 \times p'_{1i}} R'_{1i}$ for $i = 1, \ldots, n$.
3. Return the matrices $R'_{1i} F_{0i} \ldots F_{02}$ (or their reductions with respect to $D^{1 \times p_{11}} R_{11}$) for $i = 1, \ldots, n$.

Algorithm 2 is implemented in the PurityFiltration package [45].

Let us now compute a finite presentation of the left D-module M_i's. The identity $R_{1i} R_{0i} = 0$ yields $D^{1 \times p_{1i}} R_{1i} \subseteq \ker_D(.R_{0i}) = D^{1 \times p'_{1i}} R'_{1i}$, and thus there exists $R''_{1i} \in D^{p_{1i} \times p'_{1i}}$ such that:

$$\forall\, i = 1, \ldots, n, \quad R_{1i} = R''_{1i} R'_{1i}. \tag{63}$$

Applying Proposition 4 to the left D-module T_i, we obtain

$$\begin{aligned} \forall\, i = 1, \ldots, n, \quad T_i = \ker_D(.R_{0i})/\mathrm{im}_D(.R_{1i}) &= (D^{1 \times p'_{1i}} R'_{1i})/(D^{1 \times p_{1i}} R_{1i}) \\ &\cong L_i \triangleq D^{1 \times p'_{1i}}/(D^{1 \times p_{1i}} R''_{1i} + D^{1 \times p'_{2i}} R'_{2i}), \end{aligned} \tag{64}$$

where the above left D-isomorphism χ_i is defined by

$$\begin{array}{rcl} L_i = D^{1 \times p'_{1i}}/(D^{1 \times p_{1i}} R''_{1i} + D^{1 \times p'_{2i}} R'_{2i}) & \xrightarrow{\chi_i} & T_i = (D^{1 \times p'_{1i}} R'_{1i})/(D^{1 \times p_{1i}} R_{1i}) \\ \rho'_i(\lambda) & \longmapsto & \rho_i(\lambda\, R'_{1i}), \end{array} \tag{65}$$

and $\rho'_i \colon D^{1 \times p'_{1i}} \longrightarrow L_i$ is the canonical projection onto the left D-module L_i. The inverse $\chi_i^{-1} \in \hom_D(T_i, L_i)$ is then defined by $\chi_i^{-1}(\rho_i(\lambda\, R'_{1i})) = \rho'_i(\lambda)$ for all $\lambda \in D^{1 \times p'_{1i}}$.

Let us complete the commutative diagram (68) to determine the left D-homomorphism $\overline{\gamma}_{(i+1)i}$ induced by the left D-homomorphism $\gamma_{(i+1)i}$ and the left D-isomorphisms χ_i and χ_{i+1}. Using (30) with $k = j = i$ and $i = 2, \ldots, n$, we obtain $F_{0i} R_{0(i-1)} = R_{0i} F_{-1i}$. Pre-multiplying this identity by R'_{1i}, we get $R'_{1i} F_{0i} R_{0(i-1)} = R'_{1i} R_{0i} F_{-1i} = 0$, and thus $D^{1 \times p'_{1i}} (R'_{1i} F_{0i}) \subseteq \ker_D(.R_{0(i-1)}) = D^{1 \times p'_{1(i-1)}} R'_{1(i-1)}$, which proves the existence of $F'_{1i} \in D^{p'_{1i} \times p'_{1(i-1)}}$ such that:

$$\forall\, i = 2, \ldots, n, \quad R'_{1i} F_{0i} = F'_{1i} R'_{1(i-1)}. \tag{66}$$

Similarly, we can prove the existence of a matrix $F'_{2i} \in D^{p'_{2i} \times p'_{2(i-1)}}$ such that:

$$\forall\, i = 2, \ldots, n, \quad R'_{2i} F'_{1i} = F'_{2i} R'_{2(i-1)}. \tag{67}$$

Thus, the commutative diagram (69) formed by horizontal exact sequences holds.

Let us now deduce identities which will be used in what follows. Combining (28), (29), (30), (63) and (66), for $i = 1, \ldots, n$, we get

$$\begin{aligned} F_{1(i+1)} (R''_{1i} R'_{1i}) = F_{1(i+1)} R_{1i} = R_{1(i+1)} F_{0(i+1)} &= (R''_{1(i+1)} R'_{1(i+1)}) F_{0(i+1)} \\ &= R''_{1(i+1)} F'_{1(i+1)} R'_{1i}, \end{aligned}$$

$$
\begin{tikzcd}
D^{1\times p_{-1n}} \arrow[d, ".F_{-1n}"] & D^{1\times p_{0n}} \arrow[l, ".R_{0n}"'] \arrow[d, ".F_{0n}"] & D^{1\times p'_{1n}} \arrow[l, ".R'_{1n}"'] & D^{1\times p'_{2n}} \arrow[l, ".R'_{2n}"'] \\
D^{1\times p_{-1(n-1)}} \arrow[d, ".F_{-1(n-1)}"] & D^{1\times p_{0(n-1)}} \arrow[l, ".R_{0(n-1)}"'] \arrow[d, ".F_{0(n-1)}"] & D^{1\times p'_{1(n-1)}} \arrow[l, ".R'_{1(n-1)}"'] & D^{1\times p'_{2(n-1)}} \arrow[l, ".R'_{2(n-1)}"'] \\
\vdots & \vdots & \vdots & \vdots \\
\arrow[d, ".F_{-14}"] & \arrow[d, ".F_{04}"] & & \\
D^{1\times p_{-13}} \arrow[d, ".F_{-13}"] & D^{1\times p_{03}} \arrow[l, ".R_{03}"'] \arrow[d, ".F_{03}"] & D^{1\times p'_{13}} \arrow[l, ".R'_{13}"'] & D^{1\times p'_{23}} \arrow[l, ".R'_{23}"'] \\
D^{1\times p_{-12}} \arrow[d, ".F_{-12}"] & D^{1\times p_{02}} \arrow[l, ".R_{02}"'] \arrow[d, ".F_{02}"] & D^{1\times p'_{12}} \arrow[l, ".R'_{12}"'] & D^{1\times p'_{22}} \arrow[l, ".R'_{22}"'] \\
D^{1\times p_{-11}} & D^{1\times p_{01}} \arrow[l, ".R_{01}"'] & D^{1\times p'_{11}} \arrow[l, ".R'_{11}"'] & D^{1\times p'_{21}} \arrow[l, ".R'_{21}"'] .
\end{tikzcd}
\tag{68}
$$

$$
\begin{tikzcd}
D^{1\times p_{-1n}} \arrow[d, ".F_{-1n}"] & D^{1\times p_{0n}} \arrow[l, ".R_{0n}"'] \arrow[d, ".F_{0n}"] & D^{1\times p'_{1n}} \arrow[l, ".R'_{1n}"'] \arrow[d, ".F'_{1n}"] & D^{1\times p'_{2n}} \arrow[l, ".R'_{2n}"'] \arrow[d, ".F'_{2n}"] \\
D^{1\times p_{-1(n-1)}} \arrow[d, ".F_{-1(n-1)}"] & D^{1\times p_{0(n-1)}} \arrow[l, ".R_{0(n-1)}"'] \arrow[d, ".F_{0(n-1)}"] & D^{1\times p'_{1(n-1)}} \arrow[l, ".R'_{1(n-1)}"'] \arrow[d, ".F'_{1(n-1)}"] & D^{1\times p'_{2(n-1)}} \arrow[l, ".R'_{2(n-1)}"'] \arrow[d, ".F'_{2(n-1)}"] \\
\vdots & \vdots & \vdots & \vdots \\
\arrow[d, ".F_{-14}"] & \arrow[d, ".F_{04}"] & \arrow[d, ".F'_{14}"] & \arrow[d, ".F'_{24}"] \\
D^{1\times p_{-13}} \arrow[d, ".F_{-13}"] & D^{1\times p_{03}} \arrow[l, ".R_{03}"'] \arrow[d, ".F_{03}"] & D^{1\times p'_{13}} \arrow[l, ".R'_{13}"'] \arrow[d, ".F'_{13}"] & D^{1\times p'_{23}} \arrow[l, ".R'_{23}"'] \arrow[d, ".F'_{23}"] \\
D^{1\times p_{-12}} \arrow[d, ".F_{-12}"] & D^{1\times p_{02}} \arrow[l, ".R_{02}"'] \arrow[d, ".F_{02}"] & D^{1\times p'_{12}} \arrow[l, ".R'_{12}"'] \arrow[d, ".F'_{12}"] & D^{1\times p'_{22}} \arrow[l, ".R'_{22}"'] \arrow[d, ".F'_{22}"] \\
D^{1\times p_{-11}} & D^{1\times p_{01}} \arrow[l, ".R_{01}"'] & D^{1\times p'_{11}} \arrow[l, ".R'_{11}"'] & D^{1\times p'_{21}} \arrow[l, ".R'_{21}"'] .
\end{tikzcd}
\tag{69}
$$

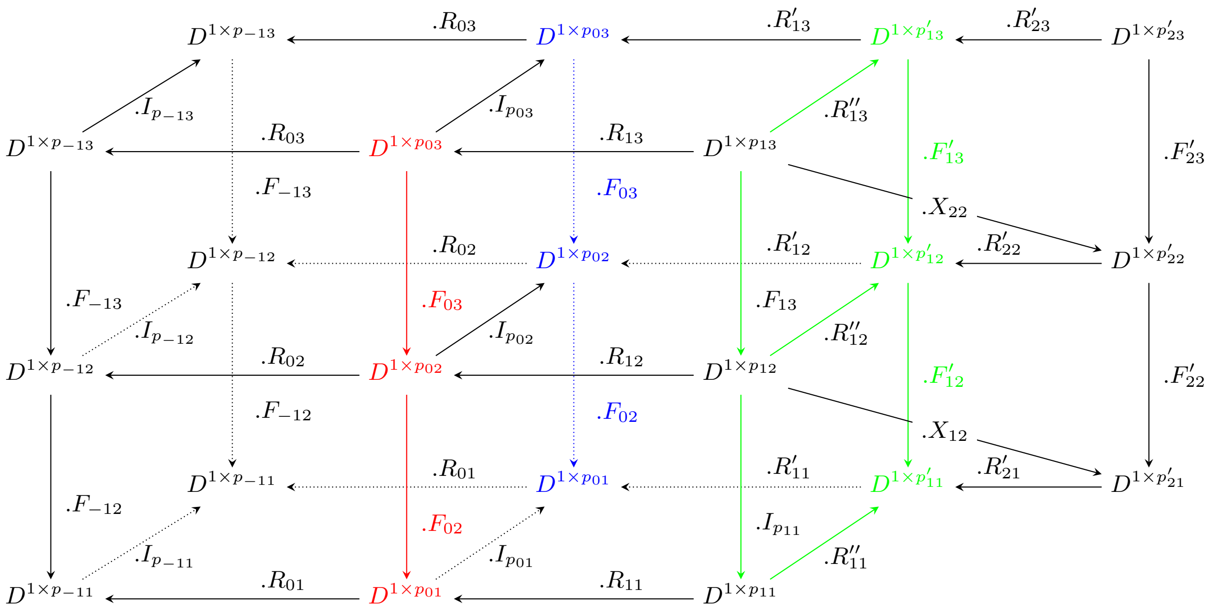

Figure 1: Bottom part of the main diagram defining the grade filtration of M

and thus $(F_{1(i+1)}\, R''_{1i} - R''_{1(i+1)}\, F'_{1(i+1)})\, R'_{1i} = 0$, i.e.,

$$D^{1\times p_{1(i+1)}}\,(F_{1(i+1)}\, R''_{1i} - R''_{1(i+1)}\, F'_{1(i+1)}) \subseteq \ker_D(.R'_{1i}) = D^{1\times p'_{2i}}\, R'_{2i},$$

which proves the existence of a matrix $X_{i2} \in D^{p_{1(i+1)}\times p'_{2i}}$ such that:

$$\forall\, i = 1, \ldots, n-1, \quad F_{1(i+1)}\, R''_{1i} - R''_{1(i+1)}\, F'_{1(i+1)} = X_{i2}\, R'_{2i}. \tag{70}$$

Now, $\gamma_{(i+1)i} \in \hom_D(T_{i+1}, T_i)$ then gives rise to $\overline{\gamma}_{(i+1)i} \in \hom_D(L_{i+1}, L_i)$ defined by

$$\forall\, i = 1, \ldots, n-1, \quad \overline{\gamma}_{(i+1)i} = \chi_i^{-1} \circ \gamma_{(i+1)i} \circ \chi_{i+1}, \tag{71}$$

where the χ_i's are defined by (65) and $\gamma_{(i+1)i}$ is defined by (48). Using (66), we get

$$\begin{aligned} \overline{\gamma}_{(i+1)i}(\rho'_{(i+1)}(\lambda)) &= (\chi_i^{-1} \circ \gamma_{(i+1)i})(\rho_{i+1}(\lambda\, R'_{1(i+1)})) = \chi_i^{-1}(\rho_i(\lambda\, R'_{1(i+1)}\, F_{0(i+1)})) \\ &= \chi_i^{-1}(\rho_i(\lambda\, F'_{1(i+1)}\, R'_{1i})) = \rho'_i(\lambda\, F'_{1(i+1)}), \end{aligned} \tag{72}$$

for all $\lambda \in D^{1\times p'_{1(i+1)}}$. Moreover, using (67) and (70), for $i = 1, \ldots, n-1$, we obtain

$$\left(\begin{array}{c} R''_{1(i+1)} \\ R'_{2(i+1)} \end{array}\right) F'_{1(i+1)} = \left(\begin{array}{c} F_{1(i+1)}\, R''_{1i} - X_{i2}\, R'_{2i} \\ F'_{2(i+1)}\, R'_{2i} \end{array}\right) = \left(\begin{array}{cc} F_{1(i+1)} & -X_{i2} \\ 0 & F'_{2(i+1)} \end{array}\right) \left(\begin{array}{c} R''_{1i} \\ R'_{2i} \end{array}\right), \tag{73}$$

which yields the following commutative exact diagram

$$\begin{tikzcd} D^{1\times(p_{1(i+1)}+p'_{2(i+1)})} \arrow[r, ".(R''^{T}_{1(i+1)} \quad R'^{T}_{2(i+1)})^T"] \arrow[d, ".G'_{1(i+1)}"] & D^{1\times p'_{1(i+1)}} \arrow[r, "\rho'_{i+1}"] \arrow[d, ".F'_{1(i+1)}"] & L_{i+1} \arrow[r] \arrow[d, "\overline{\gamma}_{(i+1)i}"] & 0 \\ D^{1\times(p_{1i}+p'_{2i})} \arrow[r, ".(R''^{T}_{1i} \quad R'^{T}_{2i})^T"] & D^{1\times p'_{1i}} \arrow[r, "\rho'_i"] & L_i \arrow[r] & 0, \end{tikzcd} \tag{74}$$

where $G'_{1(i+1)} \in D^{(p_{1(i+1)}+p'_{2(i+1)})\times(p_{1i}+p'_{2i})}$ is the first matrix appearing in the last equality of (73).

The identities $R_{11} = R''_{11}\, R'_{11}$ (see (63)) and $R'_{21}\, R'_{11} = 0$ yield the following commutative exact diagram

$$\begin{tikzcd}
D^{1\times(p_{11}+p'_{21})} \arrow[r, ".(R''^T_{11} \quad R'^T_{21})^T"] \arrow[d, ".\left(\begin{smallmatrix} I_{p_{11}} \\ 0 \end{smallmatrix}\right)"] & D^{1\times p'_{11}} \arrow[r, "\rho'_1"] \arrow[d, ".R'_{11}"] & L_1 \arrow[r] \arrow[d, "\overline{\gamma}_{10} = \gamma_{10}\circ\chi_1"] & 0 \\
D^{1\times p_{11}} \arrow[r, ".R_{11}"] & D^{1\times p_{01}} \arrow[r, "\pi"] & M \arrow[r] & 0,
\end{tikzcd} \tag{75}$$

where $\overline{\gamma}_{10} = \gamma_{10}\circ\chi_1 \in \hom_D(L_1, M)$ is defined by:

$$\forall\, \lambda \in D^{1\times p'_{11}}, \quad \overline{\gamma}_{10}(\rho'_1(\lambda)) = \pi(\lambda\, R'_{11}). \tag{76}$$

The matrices previously introduced can be rearranged into the three dimensional diagram whose bottom part is shown in Figure 1. Each two dimensional diagram of Figure 1 commutes except for the two diagrams marked in green ("faces in the depth direction") (see (70)). The horizontal sequences are either complexes (marked in red) or are exact sequences (marked in blue and in green). The vertical sequences are not complexes. The defect of exactness $T_i = \mathrm{ext}^i_D(N_{ii}, D)$ of the i^{th} horizontal complex at $D^{1\times p_{0i}}$ (marked in red) is isomorphic to the cokernel L_i of the left D-homomorphism $D^{1\times(p_{1i}+p'_{2i})} \longrightarrow D^{1\times p'_{1i}}$ defined by the two left D-homomorphisms $.R''_{1i}\colon D^{1\times p_{1i}} \longrightarrow D^{1\times p'_{1i}}$ and $.R'_{2i}\colon D^{1\times p'_{2i}} \longrightarrow D^{1\times p'_{1i}}$ arriving at $D^{1\times p'_{1i}}$ (marked in green), i.e., $L_i = D^{1\times p'_{1i}}/(D^{1\times(p_{1i}+p'_{2i})}\,(R''^T_{1i} \quad R'^T_{2i})^T)$. The left D-homomorphism $\gamma_{i(i-1)}\colon T_i \longrightarrow T_{i-1}$ defined by (48), i.e., by means of the left D-homomorphism $.F_{0i}$ (marked in red), induces $\overline{\gamma}_{i(i-1)} \in \hom_D(L_i, L_{i-1})$ defined by (72), i.e., by means of the left D-homomorphism $.F'_{1i}$ (marked in green).

Algorithm 3. • **Input:** A noetherian regular ring D satisfying (38), $\mathrm{gld}(D) = n$, and $R \in D^{q\times p}$.

- **Output:** A sequence $\{L_i\}_{i=1,\ldots,n}$ of finitely presented left D-modules and a sequence $\{\overline{\gamma}_{10} \in \hom_D(L_1, M)\} \cup \{\overline{\gamma}_{(i+1)i} \in \hom_D(L_{i+1}, L_i)\}_{i=1,\ldots,n-1}$ of left D-homomorphisms defined by (65).

1. Apply Algorithm 2 to D and $R \in D^{q\times p}$ to get matrices $R_{0i} \in D^{p_{0i}\times p_{-1i}}$ for $i = 1, \ldots, n$, matrices $F_{0i} \in D^{p_{0i}\times p_{0(i-1)}}$ for $i = 2, \ldots, n$, and matrices $R'_{1i} \in D^{p'_{1i}\times p_{0i}}$ such that $\ker_D(.R_{0i}) = D^{1\times p'_{1i}}\, R'_{1i}$ for $i = 1, \ldots, n$.
2. Compute $R'_{2i} \in D^{p'_{2i}\times p'_{1i}}$ such that $\ker_D(.R'_{1i}) = D^{1\times p'_{2i}}\, R'_{2i}$ for $i = 1, \ldots, n$.
3. Left factorize R_{1i} by R'_{1i} to get $R''_{1i} \in D^{p_{1i}\times p'_{1i}}$ such that $R_{1i} = R''_{1i}\, R'_{1i}$ for $i = 1, \ldots, n$.
4. Compute $F'_{1i} \in D^{p'_{1i}\times p'_{1(i-1)}}$ such that $R'_{1i}\, F_{0i} = F'_{1i}\, R'_{1(i-1)}$ for $i = 2, \ldots, n$.
5. Return the left D-modules $L_i = D^{1\times p'_{1i}}/(D^{1\times(p_{1i}+p'_{2i})}\,(R''^T_{1i} \quad R'^T_{2i})^T)$ for $i = 1, \ldots, n$, the matrix R'_{11} which defines $\overline{\gamma}_{10} \in \hom_D(L_1, M)$ defined by (76), and the matrices $F'_{1(i+1)}$ which define $\overline{\gamma}_{(i+1)i} \in \hom_D(L_{i+1}, L_i)$ by (72) for $i = 1, \ldots, n-1$.

Algorithm 3 is implemented in the PurityFiltration package [45].

Using 3 of Proposition 5, we obtain the following explicit finite presentation of $\operatorname{coker} \overline{\gamma}_{(i+1)i}$:

$$\operatorname{coker} \overline{\gamma}_{(i+1)i} = D^{1\times p'_{1i}}/(D^{1\times p'_{1i}}\, F'_{1i} + D^{1\times p_{1i}}\, R''_{1i} + D^{1\times p'_{2i}}\, R'_{2i}), \quad i = 1, \ldots, n-1. \tag{77}$$

We shall denote by $\sigma_i : D^{1\times p'_{1i}} \longrightarrow \operatorname{coker} \overline{\gamma}_{(i+1)i}$ the canonical projection onto $\operatorname{coker} \overline{\gamma}_{(i+1)i}$.

Up to isomorphism, the short exact sequences

$$0 \longrightarrow T_{i+1} \xrightarrow{\gamma_{(i+1)i}} T_i \longrightarrow \operatorname{coker} \gamma_{(i+1)i} \longrightarrow 0, \quad i = 1, \ldots, n-1,$$

defined in (53) (see also (46)) give rise to the following exact sequences:

$$0 \longrightarrow L_{i+1} \xrightarrow{\overline{\gamma}_{(i+1)i}} L_i \xrightarrow{\theta_i} \operatorname{coker} \overline{\gamma}_{(i+1)i} \longrightarrow 0, \quad i = 1, \ldots, n-1. \tag{78}$$

Since both γ_{10} and χ_1 are injective so is $\overline{\gamma}_{10}$, and (75) yields the following short exact sequence

$$0 \longrightarrow L_1 \xrightarrow{\overline{\gamma}_{10}} M \xrightarrow{\rho} M/M_1 \longrightarrow 0, \tag{79}$$

where $M/M_1 \cong D^{1\times p_{01}}/\ker_D(.R_{01}) = D^{1\times p_{01}}/(D^{1\times p'_{11}}\, R'_{11})$ (see (50)).

We recall that $\operatorname{coker} \overline{\gamma}_{(i+1)i} \cong \operatorname{coker} \gamma_{(i+1)i} \subseteq \operatorname{ext}^i_D(\operatorname{ext}^i_D(M, D), D)$ (see (54)), and thus $\operatorname{coker} \overline{\gamma}_{(i+1)i}$ is either zero or an i-pure left D-module when D is an Auslander regular ring (see 3 of Theorem 8 and Remark 7). Exact sequences (78) and (79) will be used in Section 4.

Remark 15. Let us point out that the left D-modules M_i's can also be characterized by means of the left D-homomorphisms $\overline{\gamma}_{i(i-1)}$'s. Combining (74) with (75), we obtain the following commutative exact diagram:

$$\begin{tikzcd}
D^{1\times(p_{1i}+p'_{2i})} \arrow[r, ".(R''^T_{1i} \quad R'^T_{2i})^T"] \arrow[d, ".\left(G'_{1i} \ldots G'_{12} \begin{pmatrix} I_{p_{11}} \\ 0 \end{pmatrix}\right)"'] & D^{1\times p'_{1i}} \arrow[r, "\rho'_i"] \arrow[d, ".(F'_{1i} \ldots F'_{12}\, R'_{11})"] & L_i \arrow[r] \arrow[d, "\overline{\gamma}_{10} \circ \overline{\gamma}_{21} \circ \ldots \circ \overline{\gamma}_{i(i-1)}"] & 0 \\
D^{1\times p_{11}} \arrow[r, ".R_{11}"] & D^{1\times p_{01}} \arrow[r, "\pi"] & M \arrow[r] & 0.
\end{tikzcd}$$

By construction (see (66)), the identity $R'_{1i}\, F_{1i} \ldots F_{12} = F'_{1i} \ldots F'_{12}\, R'_{11}$ holds. Hence, using (62) and 2 of Proposition 5, we obtain:

$$\operatorname{im}(\overline{\gamma}_{10} \circ \overline{\gamma}_{21} \circ \ldots \circ \overline{\gamma}_{i(i-1)}) = (D^{1\times p'_{1i}}\, (F'_{1i} \ldots F'_{12}\, R'_{11}) + D^{1\times p_{11}}\, R_{11})/(D^{1\times p_{11}}\, R_{11}) = M_i.$$

Hence, the residue classes of the rows of the matrix $R'_{1i}\, F_{1i} \ldots F_{12} = F'_{1i} \ldots F'_{12}\, R'_{11}$ in the left D-module $M = D^{1\times p_{01}}/(D^{1\times p_{11}}\, R_{11})$ generates the left D-module M_i for $i = 1, \ldots, n$.

Finally, we explain an efficient way to obtain the grade filtration of a nontrivial $\operatorname{ext}^i_D(N, D)$ for $i \geq 1$. We consider the case of a right D-module N (the case of a left D-module is similar). Let us first study the case of $\operatorname{ext}^1_D(N, D)$, where $N = D^q/(R\, D^p)$. If we introduce the Auslander transpose $M = D^{1\times p}/(D^{1\times q}\, R)$ of N, then the above results shows that $t_1(M) = \operatorname{ext}^1_D(N, D)$, and thus the grade filtration of $\operatorname{ext}^1_D(N, D)$ can be obtained by computing the grade filtration of M. Let us now study the case $i \geq 2$. Considering a free resolution (4) of N and introducing the right D-module $P = D^{q_{i-1}}/(S_i\, D^{q_i}) \cong \operatorname{im}_D(S_{i-1}.)$, then applying Proposition 1 to the long exact sequence $0 \longleftarrow N \xleftarrow{\kappa} D^{q_0} \xleftarrow{S_1.} D^{q_1} \xleftarrow{S_2.} \ldots \xleftarrow{S_{i-2}.} D^{q_{i-2}} \longleftarrow P \longleftarrow 0$, we get $\operatorname{ext}^i_D(N, D) \cong \operatorname{ext}^1_D(P, D) = t_1(L)$, where $L = D^{1\times q_i}/(D^{1\times p_{i-1}}\, S_i)$ is the Auslander transpose of P, which shows that the grade filtration of L gives the grade filtration of $\operatorname{ext}^i_D(N, D)$. The corresponding algorithm is implemented in the PurityFiltration package [45].

4 Equidimensional triangularization of linear systems

The purpose of this section is to apply Theorem 7 on Baer's extensions to the short exact sequences (78) and (79) to obtain a block-triangular matrix which presents the finitely generated left D-module M, and whose block-diagonal matrices are presentation matrices of the pure left D-modules M_i/M_{i+1}, where the M_i's are the left D-modules defined by the filtration (57) of M.

To simplify the exposition, we only consider the first three terms of the filtration (57) of M, namely, $M_3 \subseteq M_2 \subseteq M_1 \subseteq M$, to obtain a presentation matrix P of M based on the presentation matrices of the left D-modules M_3, M_2/M_3, M_1/M_2 and M/M_1. If D is an Auslander regular ring, then M/M_1 (resp., M_1/M_2, M_2/M_3) is 0-pure (resp., 1-pure, 2-pure). The left D-module M_3 satisfies $j_D(M_3) \geq 3$ but it is generally not 3-pure (it is the case if $\mathrm{gld}(D) = 3$). But, from the clear pattern of the presentation matrix P, we can easily determine the general result.

We point out that the approach used here emphasizes another main advantage of our approach over the ones based on more sophisticated techniques of homological algebra. If we do no want to separate the elements of M of grade number greater than or equal to j, then we only need to compute the first j terms of the filtration (57) of M. But, it does not seem to be easy to stop a spectral sequence computation to only get the first steps of the grade filtration (57).

By (78) and (79), the following short exact sequences hold

$$\begin{array}{c} 0 \longrightarrow L_3 \xrightarrow{\overline{\gamma}_{32}} L_2 \xrightarrow{\theta_2} \operatorname{coker} \overline{\gamma}_{32} \longrightarrow 0, \\ 0 \longrightarrow L_2 \xrightarrow{\overline{\gamma}_{21}} L_1 \xrightarrow{\theta_1} \operatorname{coker} \overline{\gamma}_{21} \longrightarrow 0, \\ 0 \longrightarrow L_1 \xrightarrow{\overline{\gamma}_{10}} M \xrightarrow{\rho} M/M_1 \longrightarrow 0, \end{array} \tag{80}$$

where L_i (resp., $\operatorname{coker} \overline{\gamma}_{(i+1)i}$) is defined by (64) (resp., (77)) and $M/M_1 \cong D^{1\times p_{01}}/(D^{1\times p'_{11}}\, R'_{11})$.

Using the definitions of L_2, L_3, and $\operatorname{coker} \overline{\gamma}_{32}$ (see (65) and (77)), the following commutative exact diagram holds

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & D^{1\times p_{12}}\, R''_{12} + D^{1\times p'_{22}}\, R'_{22} \arrow[d] & & \\
 & D^{1\times (p'_{13}+p_{12}+p'_{22})} \arrow[r, ".(F'^T_{13} \quad R''^T_{12} \quad R'^T_{22})^T"] \arrow[d, "\psi_2"] & D^{1\times p'_{12}} \arrow[r, "\sigma_2"] \arrow[d, "\rho'_2"] & \operatorname{coker} \overline{\gamma}_{32} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & L_3 \arrow[r, "\overline{\gamma}_{32}"] & L_2 \arrow[r, "\theta_2"] \arrow[d] & \operatorname{coker} \overline{\gamma}_{32} \arrow[r] & 0, \\
 & & 0 & &
\end{tikzcd}$$

where $\psi_2 \colon D^{1\times (p'_{13}+p_{12}+p'_{22})} \longrightarrow L_3$ is the left D-homomorphism defined by:

$$\psi_2(e_i) = \begin{cases} \rho'_3(e_i) & i = 1, \ldots, p'_{13}, \\ 0, & i = p'_{13}+1, \ldots, p'_{13}+p_{12}+p'_{22}. \end{cases}$$

Applying Theorem 7 to the first short exact sequence of (80) with the matrix

$$A = (I_{p'_{13}}^T \quad 0^T \quad 0^T)^T \in D^{(p'_{13}+p_{12}+p'_{22})\times p'_{13}}$$

(see Corollary 2), we obtain the following characterization of the left D-module L_2 in terms of the presentations of the left D-modules L_3 and $\operatorname{coker} \overline{\gamma}_{32}$.

Proposition 9. *With the previous hypotheses and notations, let us consider*

$$Q_2 = \begin{pmatrix} R''_{12} \\ R'_{22} \end{pmatrix} \in D^{(p_{12}+p'_{22})\times p'_{12}}, \quad P_2 = \begin{pmatrix} F'_{13} & -I_{p'_{13}} \\ R''_{12} & 0 \\ R'_{22} & 0 \\ 0 & R''_{13} \\ 0 & R'_{23} \end{pmatrix} \in D^{(p'_{13}+p_{12}+p'_{22}+p_{13}+p'_{23})\times(p'_{12}+p'_{13})},$$

and the following two finitely presented left D-modules:

$$\begin{cases} L_2 = D^{1\times p'_{12}}/(D^{1\times p_{12}}\, R''_{12} + D^{1\times p'_{22}}\, R'_{22}), \\ E_2 = D^{1\times(p'_{12}+p'_{13})}/(D^{1\times(p'_{13}+p_{12}+p'_{22}+p_{13}+p'_{23})}\, P_2). \end{cases}$$

If $\varrho_2 : D^{1\times(p'_{12}+p'_{13})} \longrightarrow E_2$ is the canonical projection onto E_2, then we have $E_2 \cong L_2$, where the left D-isomorphism is defined by:

$$\begin{array}{rcl} \phi_2 \colon L_2 & \longrightarrow & E_2 \\ \rho'_2(\mu) & \longmapsto & \varrho_2(\mu\,(I_{p'_{12}} \quad 0)), \end{array} \qquad \begin{array}{rcl} \phi_2^{-1} \colon E_2 & \longrightarrow & L_2 \\ \varrho_2(\nu) & \longmapsto & \rho'_2(\nu\,(I_{p'_{12}}^T \quad F'^T_{13})^T). \end{array} \tag{81}$$

Proof. Let us consider the following matrices:

$$V_2 = (I_{p'_{12}} \quad 0) \in D^{p'_{12}\times(p'_{12}+p'_{13})}, \quad W_2 = \begin{pmatrix} 0 & I_{p_{12}} & 0 & 0 & 0 \\ 0 & 0 & I_{p'_{22}} & 0 & 0 \end{pmatrix} \in D^{(p_{12}+p'_{22})\times(p'_{13}+p_{12}+p'_{22}+p_{13}+p'_{23})},$$

$$X_2 = \begin{pmatrix} I_{p'_{12}} \\ F'_{13} \end{pmatrix} \in D^{(p'_{12}+p'_{13})\times p'_{12}}, \quad Y_2 = \begin{pmatrix} 0 & 0 \\ I_{p_{12}} & 0 \\ 0 & I_{p'_{22}} \\ F_{13} & -X_{22} \\ 0 & F'_{23} \end{pmatrix} \in D^{(p'_{13}+p_{12}+p'_{22}+p_{13}+p'_{23})\times(p_{12}+p'_{22})}.$$

Using (67) and (70), we can easily check that $Q_2\, V_2 = W_2\, P_2$ (resp., $P_2\, X_2 = Y_2\, Q_2$), which by Proposition 5 induces $\phi_2 \in \hom_D(L_2, E_2)$ defined by (81) (resp., $\psi_2 \in \hom_D(E_2, L_2)$). Since $V_2\, X_2 = I_{p'_{12}}$, we get $\psi_2 \circ \phi_2 = \mathrm{id}_{L_2}$, which shows that ϕ_2 is injective. Using 3 of Proposition 5, the left D-module $\operatorname{coker} \phi_2$ is finitely presented by the matrix $(V_2^T \quad P_2^T)^T$, which admits the following left inverse over D:

$$\begin{pmatrix} I_{p'_{12}} & 0 & 0 & 0 & 0 \\ F'_{13} & -I_{p'_{13}} & 0 & 0 & 0 \end{pmatrix} \in D^{(p'_{12}+p'_{13})\times(p'_{12}+p'_{13}+p_{12}+p'_{22}+p_{13}+p'_{23})}.$$

Hence, $\operatorname{coker} \phi_2 = 0$, i.e., ϕ_2 is surjective, and thus ϕ_2 is an isomorphism, $E_2 \cong L_2$, and $\phi_2^{-1} = \psi_2$. □

Using the left D-isomorphism $\phi_2^{-1} \colon E_2 \longrightarrow L_2$ defined by (81), the second short exact sequence of (80) yields the following short exact sequence

$$0 \longrightarrow E_2 \xrightarrow{\overline{\gamma}_{21} \circ \phi_2^{-1}} L_1 \xrightarrow{\theta_1} \operatorname{coker} \overline{\gamma}_{21} \longrightarrow 0, \tag{82}$$

where using (72), the left D-homomorphism $\overline{\gamma}_{21} \circ \phi_2^{-1} : E_2 \longrightarrow L_1$ is defined by:

$$\forall\, \nu \in D^{1 \times (p'_{12} + p'_{13})}, \quad (\overline{\gamma}_{21} \circ \phi_2^{-1})(\varrho_2(\nu)) = \overline{\gamma}_{21}\left(\rho'_2\left(\nu \begin{pmatrix} I_{p'_{12}} \\ F'_{13} \end{pmatrix}\right)\right) = \rho'_1\left(\nu \begin{pmatrix} F'_{12} \\ F'_{13}\, F'_{12} \end{pmatrix}\right).$$

Using the definitions of L_1, E_2, and $\operatorname{coker} \overline{\gamma}_{21}$ (see (65), Proposition 9 and (77)), we get the commutative exact diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & D^{1 \times p_{11}}\, R''_{11} + D^{1 \times p'_{21}}\, R'_{21} \arrow[d] & & \\
 & D^{1 \times (p'_{12} + p_{11} + p'_{21})} \arrow[r, ".(F'^{T}_{12} \quad R''^{T}_{11} \quad R'^{T}_{21})^{T}"] \arrow[d, "\psi_1"] & D^{1 \times p'_{11}} \arrow[r, "\sigma_1"] \arrow[d, "\rho'_1"] & \operatorname{coker} \overline{\gamma}_{21} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & E_2 \arrow[r, "\overline{\gamma}_{21} \circ \phi_2^{-1}"] & L_1 \arrow[r, "\theta_1"] \arrow[d] & \operatorname{coker} \overline{\gamma}_{21} \arrow[r] & 0, \\
 & & 0 & &
\end{tikzcd}$$

where $\psi_1 \colon D^{1 \times (p'_{12} + p_{11} + p'_{21})} \longrightarrow E_2$ is the left D-homomorphism defined by

$$\psi_1(f_j) = \begin{cases} \varrho_2(f_j\, F), & j = 1, \ldots, p'_{12}, \\ 0, & j = p'_{12} + 1, \ldots, p'_{12} + p_{11} + p'_{21}, \end{cases}$$

$\{f_j\}_{j=1,\ldots,p'_{12}+p_{11}+p'_{21}}$ is the standard basis of $D^{1 \times (p'_{12} + p_{11} + p'_{21})}$ and:

$$F = \begin{pmatrix} I_{p'_{12}} & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \in D^{(p'_{12} + p_{11} + p'_{21}) \times (p'_{12} + p'_{13})}.$$

Applying Theorem 7 to the short exact sequence (82) with the matrix $A = F$ (see Corollary 2), we obtain the following proposition.

Proposition 10. *With the hypotheses of Proposition 9 and the previous notations, let us consider the following two matrices*

$$P_1 = \begin{pmatrix} F'_{12} & -I_{p'_{12}} & 0 \\ R''_{11} & 0 & 0 \\ R'_{21} & 0 & 0 \\ 0 & F'_{13} & -I_{p'_{13}} \\ 0 & R''_{12} & 0 \\ 0 & R'_{22} & 0 \\ 0 & 0 & R''_{13} \\ 0 & 0 & R'_{23} \end{pmatrix} \in D^{(p'_{12}+p_{11}+p'_{21}+p'_{13}+p_{12}+p'_{22}+p_{13}+p'_{23})\times(p'_{11}+p'_{12}+p'_{13})},$$
$$Q_1 = \begin{pmatrix} R''_{11} \\ R'_{21} \end{pmatrix} \in D^{(p_{11}+p'_{21})\times p'_{11}},$$

and the following two finitely presented left D-modules:

$$\begin{cases} L_1 = D^{1\times p'_{11}}/(D^{1\times(p_{11}+p'_{21})}\, Q_1), \\ E_1 = D^{1\times(p'_{11}+p'_{12}+p'_{13})}/(D^{1\times(p'_{12}+p_{11}+p'_{21}+p'_{13}+p_{12}+p'_{22}+p_{13}+p'_{23})}\, P_1). \end{cases}$$

If $\varrho_1 \colon D^{1\times(p'_{11}+p'_{12}+p'_{13})} \longrightarrow E_1$ is the canonical projection onto E_1, then we have $E_1 \cong L_1$, where the left D-isomorphism is defined by:

$$\begin{array}{rcl} \phi_1 \colon L_1 & \longrightarrow & E_1 \\ \rho'_1(\nu) & \longmapsto & \varrho_1(\nu\,(I_{p'_{11}} \quad 0 \quad 0)), \end{array} \qquad \begin{array}{rcl} \phi_1^{-1} \colon E_1 & \longrightarrow & L_1 \\ \varrho_1(\lambda) & \longmapsto & \rho'_1\left(\lambda \begin{pmatrix} I_{p'_{11}} \\ F'_{12} \\ F'_{13}\, F'_{12} \end{pmatrix}\right). \end{array} \tag{83}$$

Finally, we have $L_1 \cong M_1$, with the following left D-isomorphisms:

$$\begin{array}{rcl} \chi_1 \colon L_1 & \longrightarrow & M_1 \\ \rho'_1(\nu) & \longmapsto & \pi(\nu\, R'_{11}), \end{array} \qquad \begin{array}{rcl} \chi_1^{-1} \colon M_1 & \longrightarrow & L_1 \\ \pi(\nu\, R'_{11}) & \longmapsto & \rho'_1(\nu). \end{array}$$

Proof. Let us consider the following matrices:

$$V_1 = \begin{pmatrix} I_{p'_{11}} & 0 & 0 \end{pmatrix} \in D^{p'_{11}\times(p'_{11}+p'_{12}+p'_{13})}, \quad X_1 = (I_{p'_{11}}^T \quad F'^T_{12} \quad (F'_{13}\, F'_{12})^T)^T \in D^{(p'_{11}+p'_{12}+p'_{13})\times p'_{11}},$$
$$W_1 = \begin{pmatrix} 0 & I_{p_{11}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_{p'_{21}} & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in D^{(p_{11}+p'_{21})\times(p'_{12}+p_{11}+p'_{21}+p'_{13}+p_{12}+p'_{22}+p_{13}+p'_{23})},$$
$$Y_1 = \begin{pmatrix} 0 & 0 \\ I_{p_{11}} & 0 \\ 0 & I_{p'_{21}} \\ 0 & 0 \\ I_{p_{11}} & -X_{12} \\ 0 & F'_{22} \\ F_{13} & -F_{13}\, X_{12} - X_{22}\, F'_{22} \\ 0 & F'_{23}\, F'_{22} \end{pmatrix} \in D^{(p'_{12}+p_{11}+p'_{21}+p'_{13}+p_{12}+p'_{22}+p_{13}+p'_{23})\times(p_{11}+p'_{21})}.$$

Using (67) and (70), we can check that $Q_1\, V_1 = W_1\, P_1$ (resp., $P_1\, X_1 = Y_1\, Q_1$), which by Proposition 5 induces $\phi_1 \in \hom_D(L_1, E_1)$ defined by (83) (resp., $\psi_1 \in \hom_D(E_1, L_1)$). Since $V_1\, X_1 = I_{p'_{11}}$, we get $\psi_1 \circ \phi_1 = \mathrm{id}_{L_1}$, which shows that ϕ_1 is injective. Using 3 of Proposition 5, the left D-module $\operatorname{coker}\phi_1$ is finitely presented by the matrix $(V_1^T \quad P_1^T)^T$, which admits the following left inverse over D:

$$\begin{pmatrix} I_{p'_{11}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ F'_{12} & -I_{p'_{12}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ F'_{13}\, F'_{12} & -F'_{13} & 0 & 0 & -I_{p'_{13}} & 0 & 0 & 0 & 0 \end{pmatrix} \in D^{(p'_{11}+p'_{12}+p'_{13})\times(p'_{11}+p'_{12}+p_{11}+p'_{21}+p'_{13}+p_{12}+p'_{22}+p_{13}+p'_{23})}.$$

Hence, $\operatorname{coker}\phi_1 = 0$, i.e., ϕ_1 is surjective, and thus, ϕ_1 is an isomorphism, $E_1 \cong L_1$, and $\phi_1^{-1} = \psi_1$. Finally, the last result of Proposition 10 was already proved in Remark 15. $\square$

Using Proposition 10 and Remark 15, $\overline{\gamma}_{10} \circ \phi_1^{-1} \colon E_1 \longrightarrow M_1$ is then defined by:

$$(\chi_1 \circ \phi_1^{-1})(\varrho_1(\lambda)) = \pi\left(\lambda \begin{pmatrix} R'_{11} \\ F'_{12}\, R'_{11} \\ F'_{13}\, F'_{12}\, R'_{11} \end{pmatrix}\right).$$

Then, the third short exact sequence (80) yields the following one:

$$0 \longrightarrow E_1 \xrightarrow{\overline{\gamma}_{10} \circ \phi_1^{-1}} M \xrightarrow{\rho} M/M_1 \longrightarrow 0. \tag{84}$$

Now, we can easily check that the following commutative exact diagram holds

$$\begin{tikzcd} & D^{1\times p'_{11}} \arrow[r, ".R'_{11}"] \arrow[d, "\psi"] & D^{1\times p_{01}} \arrow[r, "\pi'"] \arrow[d, "\pi"] & M/M_1 \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & E_1 \arrow[r, "\overline{\gamma}_{10} \circ \phi_1^{-1}"] & M \arrow[r, "\rho"] & M/M_1 \arrow[r] & 0, \end{tikzcd}$$

where $\psi \colon D^{1\times p'_{11}} \longrightarrow E_1$ is defined by $\psi(g_k) = \varrho_1(g_k\,(I_{p'_{11}} \quad 0 \quad 0))$, and $\{g_k\}_{k=1,\ldots,p'_{11}}$ is the standard basis of $D^{1\times p'_{11}}$. Then, we can apply Theorem 7 to the short exact sequence (84) with $A = (I_{p'_{11}} \quad 0 \quad 0) \in D^{p'_{11}\times(p'_{11}+p'_{12}+p'_{13})}$ (see Corollary 2) to get the following theorem.

Theorem 11. *Let D be a noetherian domain which satisfies (38). With the previous notations, let us consider the following matrix*

$$P = \begin{pmatrix} R'_{11} & -I_{p'_{11}} & 0 & 0 \\ 0 & F'_{12} & -I_{p'_{12}} & 0 \\ 0 & R''_{11} & 0 & 0 \\ 0 & R'_{21} & 0 & 0 \\ 0 & 0 & F'_{13} & -I_{p'_{13}} \\ 0 & 0 & R''_{12} & 0 \\ 0 & 0 & R'_{22} & 0 \\ 0 & 0 & 0 & R''_{13} \\ 0 & 0 & 0 & R'_{23} \end{pmatrix} \in D^{(p'_{11}+p'_{12}+p_{11}+p'_{21}+p'_{13}+p_{12}+p'_{22}+p_{13}+p'_{23})\times(p_{01}+p'_{11}+p'_{12}+p'_{13})},$$

and the following two finitely presented left D-modules:

$$\begin{cases} M = D^{1\times p_{01}}/(D^{1\times p_{11}}\, R_{11}), \\ E = D^{1\times (p_{01}+p'_{11}+p'_{12}+p'_{13})}/(D^{1\times (p'_{11}+p'_{12}+p_{11}+p'_{21}+p'_{13}+p_{12}+p'_{22}+p_{13}+p'_{23})}\, P). \end{cases}$$

If $\varrho\colon D^{1\times (p_{01}+p'_{11}+p'_{12}+p'_{13})} \longrightarrow E$ is the canonical projection onto E, then we have $E \cong M$, where the left D-isomorphism is defined by:

$$\begin{array}{rcl} \phi\colon M & \longrightarrow & E \\ \pi(\lambda) & \longmapsto & \varrho(\lambda\, (I_{p_{01}} \quad 0 \quad 0 \quad 0)), \end{array} \qquad \begin{array}{rcl} \phi^{-1}\colon E & \longrightarrow & M \\ \varrho(\epsilon) & \longmapsto & \pi\left(\epsilon \begin{pmatrix} I_{p_{01}} \\ R'_{11} \\ F'_{12}\, R'_{11} \\ F'_{13}\, F'_{12}\, R'_{11} \end{pmatrix}\right). \end{array} \tag{85}$$

Proof. Let us consider the following matrices:

$$V = (I_{p_{01}} \quad 0 \quad 0 \quad 0) \in D^{p_{01}\times (p_{01}+p'_{11}+p'_{12}+p'_{13})},$$

$$W = \begin{pmatrix} R''_{11} & 0 & I_{p'_{11}} & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in D^{p_{11}\times (p'_{11}+p'_{12}+p_{11}+p'_{21}+p'_{13}+p_{12}+p'_{22}+p_{13}+p'_{23})},$$

$$X = \begin{pmatrix} I_{p_{01}} \\ R'_{11} \\ F'_{12}\, R'_{11} \\ F'_{13}\, F'_{12}\, R'_{11} \end{pmatrix} \in D^{(p_{01}+p'_{11}+p'_{12}+p'_{13})\times p_{01}},$$

$$Y = \begin{pmatrix} 0 \\ 0 \\ I_{p_{11}} \\ 0 \\ 0 \\ I_{p_{11}} \\ 0 \\ F_{13} \\ 0 \end{pmatrix} \in D^{(p'_{11}+p'_{12}+p_{11}+p'_{21}+p'_{13}+p_{12}+p'_{22}+p_{13}+p'_{23})\times p_{11}}.$$

Using (67) and (70), we can check that $R_{11}\, V = W\, P$ (resp., $P\, X = Y\, R_{11}$), which by Proposition 5 induces $\phi \in \hom_D(M, E)$ defined by (85) (resp., $\psi \in \hom_D(E, M)$). Moreover, since $V\, X = I_{p_{01}}$, we get $\psi \circ \phi = \mathrm{id}_M$, which shows that ϕ is injective. Using 3 of Proposition 5, the left D-module $\operatorname{coker} \phi$ is finitely presented by the matrix $(V^T \quad P^T)^T$, which admits the following left inverse over D:

$$\begin{pmatrix} I_{p_{01}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ R'_{11} & -I_{p'_{11}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ F'_{12}\, R'_{11} & -F'_{12} & -I_{p'_{12}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ F'_{13}\, F'_{12}\, R'_{11} & -F'_{13}\, F'_{12} & -F'_{13} & 0 & 0 & -I_{p'_{13}} & 0 & 0 & 0 & 0 \end{pmatrix}$$
$$\in D^{(p_{01}+p'_{11}+p'_{12}+p_{11}+p'_{21}+p'_{13}+p_{12}+p'_{22}+p_{13}+p'_{23})\times (p_{01}+p'_{11}+p'_{12}+p'_{13})}.$$

Hence, $\operatorname{coker} \phi = 0$, i.e., ϕ is surjective, and thus, ϕ is an isomorphism, $E \cong M$, and $\phi^{-1} = \psi$. $\square$

We note that (70) for $i = 1$ and $F_{12} = I_{p_{11}}$ yield the following identity:

$$R''_{11} = R''_{12}\, F'_{12} + X_{12}\, R'_{21}. \tag{86}$$

Since the third column of P contains R''_{12}, the third row of P containing the matrix R''_{11} can be removed. We then obtain the following straightforward corollary of Theorem 11.

Corollary 5. *With the hypotheses and the notations of Theorem 11, if*

$$Q = \begin{pmatrix} R'_{11} & -I_{p'_{11}} & 0 & 0 \\ 0 & F'_{12} & -I_{p'_{12}} & 0 \\ 0 & R'_{21} & 0 & 0 \\ 0 & 0 & F'_{13} & -I_{p'_{13}} \\ 0 & 0 & R''_{12} & 0 \\ 0 & 0 & R'_{22} & 0 \\ 0 & 0 & 0 & R''_{13} \\ 0 & 0 & 0 & R'_{23} \end{pmatrix} \in D^{(p'_{11}+p'_{12}+p'_{21}+p'_{13}+p_{12}+p'_{22}+p_{13}+p'_{23})\times(p_{01}+p'_{11}+p'_{12}+p'_{13})},$$

then we have

$$\begin{aligned} M &= D^{1\times p_{01}}/(D^{1\times p_{11}}\, R_{11}) \\ &\cong E = D^{1\times(p_{01}+p'_{11}+p'_{12}+p'_{13})}/(D^{1\times(p'_{11}+p'_{12}+p'_{21}+p'_{13}+p_{12}+p'_{22}+p_{13}+p'_{23})}\, Q), \end{aligned}$$

where the isomorphism is defined by (85).

Let $\mathcal{F}$ be a left D-module. Then, $M \cong E$ and Theorem 1 imply that $\ker_{\mathcal{F}}(R_{11}.) \cong \ker_{\mathcal{F}}(P.) = \ker_{\mathcal{F}}(Q.)$. Applying the functor $\hom_D(\,\cdot\,, \mathcal{F})$ to the diagram defined in Figure 1, we obtain the diagram of abelian groups defined in Figure 2 formed by horizontal complexes of abelian groups. More precisely, using (85) and $R = R_{11}$, we obtain the following corollary.

Corollary 6. *If D is a noetherian domain which satisfies (38), $R \in D^{q\times p}$, and $\mathcal{F}$ a left D-module, then*

$$\ker_{\mathcal{F}}(R.) \cong \ker_{\mathcal{F}}(Q.),$$

i.e., the following system equivalence holds

$$R\,\eta = 0 \quad \Leftrightarrow \quad \begin{cases} R'_{11}\,\zeta - \tau_1 = 0, \\ F'_{12}\,\tau_1 - \tau_2 = 0, \\ R'_{21}\,\tau_1 = 0, \\ F'_{13}\,\tau_2 - \tau_3 = 0, \\ R''_{12}\,\tau_2 = 0, \\ R'_{22}\,\tau_2 = 0, \\ R''_{13}\,\tau_3 = 0, \\ R'_{23}\,\tau_3 = 0, \end{cases} \tag{87}$$

under the following invertible transformations:

$$\begin{array}{ll} \gamma :\ \ker_{\mathcal{F}}(Q.) \longrightarrow \ker_{\mathcal{F}}(R.) & \gamma^{-1} :\ \ker_{\mathcal{F}}(R.) \longrightarrow \ker_{\mathcal{F}}(Q.) \\ \begin{pmatrix} \zeta \\ \tau_1 \\ \tau_2 \\ \tau_3 \end{pmatrix} \longmapsto \eta = \zeta, & \eta \longmapsto \begin{pmatrix} \zeta \\ \tau_1 \\ \tau_2 \\ \tau_3 \end{pmatrix} = \begin{pmatrix} I_{p_{01}} \\ R'_{11} \\ F'_{12}\, R'_{11} \\ F'_{13}\, F'_{12}\, R'_{11} \end{pmatrix} \eta. \end{array} \tag{88}$$

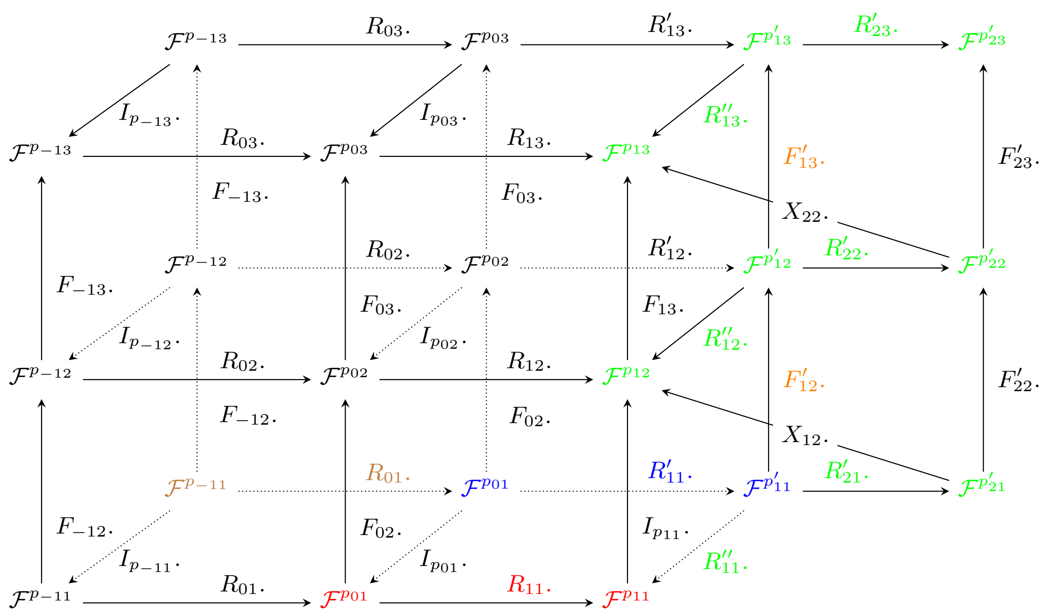

Figure 2: Dual of Figure 1

Remark 16. Let D be both an Auslander regular ring and a Cohen-Macaulay ring. If we set

$$S_0 = R'_{11}, \quad S_1 = \begin{pmatrix} F'_{12} \\ R''_{11} \\ R'_{21} \end{pmatrix}, \quad S'_1 = \begin{pmatrix} F'_{12} \\ R'_{21} \end{pmatrix}, \quad S_2 = \begin{pmatrix} F'_{13} \\ R''_{12} \\ R'_{22} \end{pmatrix}, \quad S_3 = \begin{pmatrix} R''_{13} \\ R'_{23} \end{pmatrix},$$

then:

1. $\ker_{\mathcal{F}}(S_3.) \cong \hom_D(L_3, \mathcal{F}) \cong \hom_D(\mathrm{ext}^3_D(N_{33}, D), \mathcal{F})$ is either 0 or has dimension less than or equal to $\dim(D) - 3$,

2. $\ker_{\mathcal{F}}(S_2.) \cong \hom_D(\mathrm{coker}\, \overline{\gamma}_{32}, \mathcal{F}) \cong \hom_D(\mathrm{coker}\, \gamma_{32}, \mathcal{F})$ has dimension $\dim(D) - 2$ when it is nonzero,

3. $\ker_{\mathcal{F}}(S_1.) = \ker_{\mathcal{F}}(S'_1.) \cong \hom_D(\mathrm{coker}\, \overline{\gamma}_{21}, \mathcal{F}) \cong \hom_D(\mathrm{coker}\, \gamma_{21}, \mathcal{F})$ is either 0 or has dimension $\dim(D) - 1$,

4. $\ker_{\mathcal{F}}(S_0.) \cong \hom_D(M/M_1, \mathcal{F})$ has dimension $\dim(D)$ when it is nonzero.

If R_3 has full row rank, i.e., $\ker_D(.R_3) = 0$, then $N_{33} \cong \mathrm{ext}^3_D(M, D)$, and thus $\mathrm{ext}^3_D(N_{33}, D) \cong \mathrm{ext}^3_D(\mathrm{ext}^3_D(M, D), D)$, and $\ker_{\mathcal{F}}(S_3.)$ has $\dim(D) - 3$ when it is nonzero.

The solution of the linear system $\ker_{\mathcal{F}}(R.)$ can then be obtained by integrating the linear system $\ker_{\mathcal{F}}(Q.)$, i.e., by integrating in cascade the linear system $\ker_{\mathcal{F}}(S_3.)$ of dimension less than or equal to $\dim(D) - 3$, then the inhomogeneous linear systems of dimension respectively $\dim(D) - 2$, $\dim(D) - 1$ and $\dim(D)$. Finally, if $\mathcal{F}$ is an injective left D-module, then the linear system $\ker_{\mathcal{F}}(R'_{11}.)$ of dimension $\dim(D)$ is parametrizable and $\ker_{\mathcal{F}}(R'_{11}.) = R_{01}\, \mathcal{F}^{p_{-11}}$.

Example 6. Let us consider an example studied by Janet and considered again in [38] defined by the $D = \mathbb{Q}[\partial_1, \partial_2, \partial_3]$-module $M = D^{1\times 4}/(D^{1\times 6}\, R)$ finitely presented by the following matrix:

$$R = \begin{pmatrix} 0 & -2\,\partial_1 & \partial_3 - 2\,\partial_2 - \partial_1 & -1 \\ 0 & \partial_3 - 2\,\partial_1 & 2\,\partial_2 - 3\,\partial_1 & 1 \\ \partial_3 & -6\,\partial_1 & -2\,\partial_2 - 5\,\partial_1 & -1 \\ 0 & \partial_2 - \partial_1 & \partial_2 - \partial_1 & 0 \\ \partial_2 & -\partial_1 & -\partial_2 - \partial_1 & 0 \\ \partial_1 & -\partial_1 & -2\,\partial_1 & 0 \end{pmatrix}.$$

The D-module M admits the following finite free resolution:

$$0 \longleftarrow M \xleftarrow{\pi} D^{1\times 4} \xleftarrow{.R} D^{1\times 6} \xleftarrow{.R_2} D^{1\times 4} \xleftarrow{.R_3} D \longleftarrow 0,$$

$$R_2 = \begin{pmatrix} 2\,\partial_2 & \partial_2 & -\partial_2 & -\partial_3 & \partial_3 & 0 \\ 2\,\partial_1 & \partial_2 & -2\,\partial_1 + \partial_2 & -\partial_3 & 8\,\partial_1 - \partial_3 & -8\,\partial_2 + 2\,\partial_3 \\ 0 & \partial_1 - \partial_2 & \partial_1 - \partial_2 & \partial_3 & -8\,\partial_1 + \partial_3 & 8\,\partial_2 - \partial_3 \\ 0 & 0 & 0 & \partial_1 & -\partial_1 & \partial_2 \end{pmatrix},$$

$$R_3 = (\partial_1 \quad \partial_2 \quad -\partial_2 \quad \partial_3).$$

Using the notations $R_{11} = R$, $R_{22} = R_2$, and $R_{33} = R_3$, the commutative diagram (32) becomes the following commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[r] & D \arrow[r, "R_{13}."] & D^4 \arrow[r, "R_{23}."] & D^4 \arrow[r, "R_{33}."] & D \arrow[r, "\kappa_{33}"] & N_{33} \arrow[r] & 0 \\
 & & 0 \arrow[r] & D^3 \arrow[u, "F_{03}."'] \arrow[r, "R_{12}."] & D^6 \arrow[u, "F_{13}."'] \arrow[r, "R_{22}."] & D^4 \arrow[u, equal] \arrow[r, "\kappa_{22}"] & N_{22} \arrow[r] & 0 & \\
 & 0 \arrow[r] & D \arrow[r, "R_{01}."] & D^4 \arrow[u, "F_{02}."'] \arrow[r, "R_{11}."] & D^6 \arrow[u, equal] \arrow[r, "\kappa_{11}"] & N_{11} \arrow[r] & 0 & & \\
 & & 0 \arrow[r] & D^4 \arrow[u, equal] \arrow[r, "\kappa_{00}"] & N_{00} \arrow[r] & 0, & & &
\end{tikzcd}$$

whose horizontal sequences are exact and with the following notations:

$$R_{01} = \begin{pmatrix} 1 \\ -1 \\ 1 \\ \partial_1 - 2\,\partial_2 + \partial_3 \end{pmatrix}, \quad R_{12} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 4\,\partial_1 - \partial_3 & 0 \\ 1 & 4\,\partial_1 - \partial_3 & \partial_3 \\ 0 & \partial_1 - \partial_2 & 0 \\ 0 & \partial_1 - \partial_2 & 0 \\ 0 & 0 & \partial_1 \end{pmatrix}, \quad R_{23} = \begin{pmatrix} -\partial_3 & \partial_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \partial_1 & -1 & \partial_3 \\ \partial_1 & 0 & 0 & \partial_2 \end{pmatrix},$$

$$R_{13} = \begin{pmatrix} -\partial_2 \\ -\partial_3 \\ 0 \\ \partial_1 \end{pmatrix}, \quad F_{02} = \begin{pmatrix} 0 & -2\,\partial_1 & -\partial_1 - 2\,\partial_2 + \partial_3 & -1 \\ 0 & -1 & -1 & 0 \\ 1 & -1 & -2 & 0 \end{pmatrix},$$

$$F_{13} = \begin{pmatrix} 0 & 0 & 0 & 1 & -1 & 0 \\ 2 & 1 & -1 & 0 & 0 & 0 \\ 2\,\partial_1 & \partial_2 & -2\,\partial_1 + \partial_2 & -\partial_3 & 8\,\partial_1 - \partial_3 & -8\,\partial_2 + 2\,\partial_3 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad F_{03} = (0 \quad 0 \quad 1)\,,$$

$R_{03} = 0$, and $R_{02} = 0$. Using Remark 14 with $p_{03} = 1$ and $p_{02} = 3$, we get $R'_{13} = 1$, $R'_{12} = I_3$, $R'_{21} = 0$, $R'_{22} = 0$, and $R'_{23} = 0$. Then, (69) becomes the following the commutative diagram

$$\begin{tikzcd} 0 & D \arrow[l] \arrow[d, ".F_{03}"] & D \arrow[l, ".R'_{13}"'] \arrow[d, ".F'_{13}"] & 0 \arrow[l] \\ 0 & D^{1\times 3} \arrow[l] \arrow[d, ".F_{02}"] & D^{1\times 3} \arrow[l, ".R'_{12}"'] \arrow[d, ".F'_{12}"] & 0 \arrow[l] \\ D & D^{1\times 4} \arrow[l, ".R_{01}"'] & D^{1\times 3} \arrow[l, ".R'_{11}"'] & 0, \arrow[l] \end{tikzcd}$$

with the following notations:

$$R'_{11} = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & \partial_1 - 2\,\partial_2 + \partial_3 & -1 \end{pmatrix}, \quad F'_{13} = F_{03}, \quad F'_{12} = \begin{pmatrix} 0 & -2\,\partial_1 & 1 \\ 0 & -1 & 0 \\ 1 & -1 & 0 \end{pmatrix}.$$

Moreover, using (63), we have $R''_{13} = R_{13}$, $R''_{12} = R_{12}$, and:

$$R''_{11} = \begin{pmatrix} 0 & -2\,\partial_1 & 1 \\ 0 & -2\,\partial_1 + \partial_3 & -1 \\ \partial_3 & -6\,\partial_1 & 1 \\ 0 & -\partial_1 + \partial_2 & 0 \\ \partial_2 & -\partial_1 & 0 \\ \partial_1 & -\partial_1 & 0 \end{pmatrix}.$$

Since $\ker_D(.R_3) = 0$, $N_{33} \cong \mathrm{ext}^3_D(M, D)$ and thus $\mathrm{ext}^3_D(N_{33}, D) \cong \mathrm{ext}^3_D(\mathrm{ext}^3_D(M, D), D)$, which shows that $\{M_i\}_{i=0,\ldots,3}$ defined by (57) is the grade filtration of M.

Using (45) and (64) with $N_{11} = D^6/(R_{11}\, D^4)$, $N_{22} = D^4/(R_{22}\, D^6)$, and $N_{33} = D/(R_{33}\, D^4)$, we obtain the finitely left D-modules:

$$\begin{cases} L_1 = D^{1\times 3}/(D^{1\times 6}\, R''_{11}) \cong \mathrm{ext}^1_D(N_{11}, D) \cong t(M), \\ L_2 = D^{1\times 3}/(D^{1\times 6}\, R_{12}) \cong \mathrm{ext}^2_D(N_{22}, D), \\ L_3 = D/(D^{1\times 4}\, R_{13}) \cong \mathrm{ext}^3_D(N_{33}, D). \end{cases}$$

Corollary 5 yields $M \cong E = D^{1\times 11}/(D^{1\times 17}\,Q)$, where the matrix Q is defined by:

$$Q = \left(\begin{array}{ccccccccccc}
1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & \partial_1 - 2\,\partial_2 + \partial_3 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & -2\,\partial_1 & 1 & -1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 4\,\partial_1 - \partial_3 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 4\,\partial_1 - \partial_3 & \partial_3 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \partial_1 - \partial_2 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \partial_1 - \partial_2 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \partial_1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\partial_2\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\partial_3\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \partial_1
\end{array}\right).$$

Let us explicitly compute $\ker_{\mathcal{F}}(Q.)$, where $\mathcal{F} = C^{\infty}(\mathbb{R}^3)$. We first integrate the last diagonal block of Q, i.e., the 0-dimensional (holonomic) linear system $\ker_{\mathcal{F}}(R_{13}.)$:

$$\left\{\begin{array}{l} -\partial_2\,\tau_3 = 0,\\ -\partial_3\,\tau_3 = 0,\\ \partial_1\,\tau_3 = 0, \end{array}\right. \quad \Leftrightarrow \quad \tau_3 = c_1 \in \mathbb{R}.$$

Then, we integrate the inhomogeneous linear system in $\tau_2 = (\tau_{21} \quad \tau_{22} \quad \tau_{23})^T$ and τ_3 formed by the third triangular block of Q (whose homogeneous part is purely subholonomic), namely:

$$\left\{\begin{array}{l} \tau_{23} - \tau_3 = 0,\\ \tau_{21} = 0,\\ -\tau_{21} + (4\,\partial_1 - \partial_3)\,\tau_{22} = 0,\\ \tau_{21} + (4\,\partial_1 - \partial_3)\,\tau_{22} + \partial_3\,\tau_{23} = 0,\\ (\partial_1 - \partial_2)\,\tau_{22} = 0, \end{array}\right. \quad \Leftrightarrow \quad \left\{\begin{array}{l} \tau_{23} = \tau_3 = c_1,\\ \tau_{21} = 0,\\ (4\,\partial_1 - \partial_3)\,\tau_{22} = 0,\\ (\partial_1 - \partial_2)\,\tau_{22} = 0. \end{array}\right.$$

We obtain $\tau_{21} = 0$, $\tau_{22} = f_1(x_3 + \frac{1}{4}\,(x_1 + x_2))$, where f_1 is an arbitrary smooth function, and $\tau_{23} = c_1$, where c_1 is an arbitrary constant. Then, we integrate the inhomogeneous linear system in $\tau_1 = (\tau_{11} \quad \tau_{12} \quad \tau_{13})^T$ and τ_2 formed by the second triangular block of Q, namely:

$$\left\{\begin{array}{l} -2\,\partial_1\,\tau_{12} + \tau_{13} - \tau_{21} = 0,\\ -\tau_{12} - \tau_{22} = 0,\\ \tau_{11} - \tau_{12} - \tau_{23} = 0, \end{array}\right. \quad \Leftrightarrow \quad \left\{\begin{array}{l} \tau_{12} = -\tau_{22} = -f_1(x_3 + \frac{1}{4}\,(x_1 + x_2)),\\ \tau_{11} = -\tau_{22} + \tau_{23} = -f_1(x_3 + \frac{1}{4}\,(x_1 + x_2)) + c_1,\\ \tau_{13} = -2\,\partial_1\,\tau_{22} + \tau_{21} = -\frac{1}{2}\,\dot{f}_1(x_3 + \frac{1}{4}\,(x_1 + x_2)). \end{array}\right.$$

The entries of τ_1 are 1-dimensional and not 2-dimensional. This result comes from the fact that the matrix S_1' defined in Remark 16 admits a left inverse over D. Thus, we have $M_1/M_2 = 0$, i.e., $M_1 = M_2$, which yields $\ker_{\mathcal{F}}(S_1'.) \cong \hom_D(\operatorname{coker} \overline{\gamma}_{21}, \mathcal{F}) \cong \hom_D(\operatorname{coker} \gamma_{21}, \mathcal{F}) = 0$. Finally, we integrate the inhomogeneous linear system in $\zeta = (\zeta_1 \ \ldots \ \zeta_4)^T$ and τ_1 formed by the first triangular block of P, namely:

$$\left\{ \begin{array}{l} \zeta_1 - \zeta_3 - \tau_{11} = 0, \\ \zeta_2 + \zeta_3 - \tau_{12} = 0, \\ (\partial_1 - 2\,\partial_2 + \partial_3)\,\zeta_3 - \zeta_4 - \tau_{13} = 0, \end{array} \right. \quad \Leftrightarrow \quad \left\{ \begin{array}{l} \zeta_1 - \zeta_2 = -f_1(x_3 + \frac{1}{4}\,(x_1 + x_2)) + c_1, \\ \zeta_2 + \zeta_3 = -f_1(x_3 + \frac{1}{4}\,(x_1 + x_2)), \\ (\partial_1 - 2\,\partial_2 + \partial_3)\,\zeta_3 - \zeta_4 = -\frac{1}{2}\,\dot{f}_1(x_3 + \frac{1}{4}\,(x_1 + x_2)). \end{array} \right. \tag{89}$$

The torsion-free D-module $M/t(M) = D^{1\times 4}/(D^{1\times 3}\,R_{11}')$ can be parametrized by means of R_{01}, i.e., $M/t(M) \cong D^{1\times 4}\,R_{01}$. Since $\mathcal{F}$ is an injective D-module, the linear system $\ker_{\mathcal{F}}(R_{11}'.)$ is parametrized by R_{01}, i.e., $\ker_{\mathcal{F}}(R_{11}'.) = R_{01}\,\mathcal{F}$. Since R_{11}' admits the right inverse over D

$$X = \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{array} \right),$$

the *Quillen-Suslin theorem* (see, e.g., [21, 50]) implies that $M/t(M)$ is a free D-module of rank 1. The general $\mathcal{F}$-solution of (89) is then defined by $\zeta = R_{01}\,\xi + X\,\tau_1$ (for more details, see [46]):

$$\forall\, \xi \in C^\infty(\mathbb{R}^3), \quad \forall\, f_1 \in C^\infty(\mathbb{R}), \quad \forall\, c_1 \in \mathbb{R}, \quad \left\{ \begin{array}{l} \zeta_1 = \xi - f_1(x_3 + \frac{1}{4}\,(x_1 + x_2)) + c_1, \\ \zeta_2 = -\xi - f_1(x_3 + \frac{1}{4}\,(x_1 + x_2)), \\ \zeta_3 = \xi, \\ \zeta_4 = (\partial_1 - 2\,\partial_2 + \partial_3)\,\xi + \frac{1}{2}\,\dot{f}_1(x_3 + \frac{1}{4}\,(x_1 + x_2)). \end{array} \right.$$

Finally, using the D-isomorphism γ defined by (88), we obtain

$$\left\{ \begin{array}{l} -2\,\partial_1\,\eta_2 + \partial_3\,\eta_3 - 2\,\partial_2\,\eta_3 - \partial_1\,\eta_3 - \eta_4 = 0, \\ \partial_3\,\eta_2 - 2\,\partial_1\,\eta_2 + 2\,\partial_2\,\eta_3 - 3\,\partial_1\,\eta_3 + \eta_4 = 0, \\ \partial_3\,\eta_1 - 6\,\partial_1\,\eta_2 - 2\,\partial_2\,\eta_3 - 5\,\partial_1\,\eta_3 - \eta_4 = 0, \\ \partial_2\,\eta_2 - \partial_1\,\eta_2 + \partial_2\,\eta_3 - \partial_1\,\eta_3 = 0, \\ \partial_2\,\eta_1 - \partial_1\,\eta_2 - \partial_2\,\eta_3 - \partial_1\,\eta_3 = 0, \\ \partial_1\,\eta_1 - \partial_1\,\eta_2 - 2\,\partial_1\,\eta_3 = 0, \end{array} \right. \quad \Leftrightarrow \quad \left\{ \begin{array}{l} \eta_1 = \xi - f_1(x_3 + \frac{1}{4}\,(x_1 + x_2)) + c_1, \\ \eta_2 = -\xi - f_1(x_3 + \frac{1}{4}\,(x_1 + x_2)), \\ \eta_3 = \xi, \\ \eta_4 = (\partial_1 - 2\,\partial_2 + \partial_3)\,\xi + \frac{1}{2}\,\dot{f}_1(x_3 + \frac{1}{4}\,(x_1 + x_2)), \end{array} \right. \tag{90}$$

where ξ (resp., f_1, c_1) is an arbitrary function of $C^\infty(\mathbb{R}^3)$ (resp., $C^\infty(\mathbb{R})$, constant).

For more examples coming from mathematical physics, mathematical systems theory, and algebraic geometry, see [45]. For instance, using the PurityFiltration package, we can show that the torsion submodule of the differential module M defined by the linearized Einstein equations in the vacuum (see, e.g., [14]) is 1-pure (see [45]), and thus every nontrivial torsion element m of M defines a pure differential module of dimension 3.

Using the regular patterns of the matrix P and (85), we can easily generalize Theorem 11, Corollary 6 and Remark 16 as follows.

Theorem 12. *Let D be a noetherian regular ring D satisfying (38), $\mathrm{gld}(D) = n$, and $R \in D^{q\times p}$. Then, there exists a matrix $\overline{R} \in D^{\overline{q}\times\overline{p}}$ of the form*

$$\overline{R} = \begin{pmatrix} R'_{11} & -I_{p'_{11}} & 0 & 0 & 0 & 0 \\ 0 & F'_{12} & -I_{p'_{12}} & 0 & 0 & 0 \\ 0 & R''_{11} & 0 & 0 & 0 & 0 \\ 0 & R'_{21} & 0 & 0 & 0 & 0 \\ 0 & 0 & \vdots & \vdots & 0 & 0 \\ 0 & 0 & \vdots & \vdots & 0 & 0 \\ 0 & 0 & 0 & 0 & F'_{1(n-1)} & -I_{p'_{1(n-1)}} \\ 0 & 0 & 0 & 0 & R''_{1(n-1)} & 0 \\ 0 & 0 & 0 & 0 & R'_{2(n-1)} & 0 \\ 0 & 0 & 0 & 0 & 0 & R''_{1n} \\ 0 & 0 & 0 & 0 & 0 & R'_{2n} \end{pmatrix}$$

such that $M = D^{1\times p}/(D^{1\times q}\,R) \cong \overline{M} = D^{1\times\overline{p}}/(D^{1\times\overline{q}}\,\overline{R})$. Moreover, if $\overline{\pi} : D^{1\times\overline{p}} \longrightarrow \overline{M}$ is the canonical projection onto $\overline{M}$ and $R'_{11} \in D^{p'_{11}\times p_{01}}$, then there exist matrices F'_{1i} for $i = 2, \ldots, n$ such that:

$$\begin{array}{rcl} \varphi\colon M & \longrightarrow & \overline{M} \\ \pi(\lambda) & \longmapsto & \overline{\pi}(\lambda\,(I_{p_{01}} \quad 0 \quad \cdots \quad 0)), \end{array} \qquad \begin{array}{rcl} \varphi^{-1}\colon \overline{M} & \longrightarrow & M \\ \overline{\pi}(\mu) & \longmapsto & \pi\left(\mu \begin{pmatrix} I_{p_{01}} \\ R'_{11} \\ F'_{12}\,R'_{11} \\ \vdots \\ F'_{1n}\cdots F'_{12}\,R'_{11} \end{pmatrix}\right). \end{array}$$

If $\mathcal{F}$ is a left D-module, then $\ker_{\mathcal{F}}(R.) \cong \ker_{\mathcal{F}}(\overline{R}.)$, where:

$$\begin{array}{rcl} \overline{\gamma}\colon \ker_{\mathcal{F}}(\overline{R}.) & \longrightarrow & \ker_{\mathcal{F}}(R.) \\ \begin{pmatrix} \zeta \\ \tau_1 \\ \vdots \\ \tau_n \end{pmatrix} & \longmapsto & \eta = \zeta, \end{array} \qquad \begin{array}{rcl} \overline{\gamma}^{-1}\colon \ker_{\mathcal{F}}(R.) & \longrightarrow & \ker_{\mathcal{F}}(\overline{R}.) \\ \eta & \longmapsto & \begin{pmatrix} \zeta \\ \tau_1 \\ \vdots \\ \tau_n \end{pmatrix} = \begin{pmatrix} I_{p_{01}} \\ R'_{11} \\ \vdots \\ F'_{1n}\cdots F'_{12}\,R'_{11} \end{pmatrix} \eta. \end{array}$$

Finally, if D is an Auslander regular ring, then the grade filtration $\{M_i\}_{i=0,\ldots,n}$ of M is defined by the left D-module M_i finitely presented by $(R''^{T}_{1i} \quad R'^{T}_{2i})^T$, and M_i/M_{i+1} is the i-pure left D-module finitely presented by R'_{11} for i=0, by $(F'^{T}_{1i} \quad R''^{T}_{1i} \quad R'^{T}_{2i})^T$ for $i = 1, \ldots, n-1$, and by $(R''^{T}_{1n} \quad R'^{T}_{2n})^T$ for $i = n$.

Remark 17. We note that $M_i = M_{i+1}$ iff $S_i = (F'^{T}_{1i} \quad R''^{T}_{1i} \quad R'^{T}_{2i})^T$ admits a left inverse over D. It shows that the matrix $\overline{R}$ can sometimes be simplified especially if Gröbner/Janet bases can be computed over D, since the matrix S_i does not generally form a Gröbner/Janet basis. Moreover, elementary operations can also be applied to simplify the matrix S_i (see, e.g., Example 6). Using inductively Proposition 6, we can then obtain a simple presentation matrix of M with a triangular-block form and whose diagonal blocks present the left D-modules M_i/M_{i+1}'s when they are nontrivial. Such a procedure is implemented in the PurityFiltration package. For related results, see Appendix A of [2]. Finally, if D is a commutative polynomial ring, then Remark 4 can also be used to check whether or not $M_i \cong M_i/M_{i+1} \oplus M_{i+1}$, i.e., whether or not the corresponding matrix $(I_{p'_i}^T \quad 0^T \quad 0^T)^T$ can be replaced by the trivial matrix $(0^T \quad 0^T \quad 0^T)^T$ (which generally helps the integration of the corresponding linear functional system).

Even if the size of the matrix $\overline{R}$ is larger than the one of R, the presentation matrix $\overline{R}$ is more tractable for a fine study of the module properties of the left D-module $M \cong \overline{M}$ than R, for the study of the structural properties of $\ker_{\mathcal{F}}(R.)$, as well as for computing closed-form solutions of $\ker_{\mathcal{F}}(R.)$ (when they exist). For instance, overdetermined/underdetermined linear PD systems $\ker_{\mathcal{F}}(R.)$, which cannot be directly integrated by means of standard computer algebra systems such as Maple, can be done using their equivalent forms $\ker_{\mathcal{F}}(\overline{R}.)$. See Appendix and [45].

5 An embedding theorem

If D is a domain, then a torsion-free left D-module M can be embedded into a free left D-module (see the comment after Proposition 4), and thus into a projective left D-module. Using Example 4, we deduce that a 0-pure left D-module M can be embedded into a left D-module of projective dimension 0. This result is a particular case of the following general result.

Proposition 11 ([10]). *Let D be an Auslander regular ring and M an i-pure left D-module. Then, M can be embedded into a left D-module P_i of left projective dimension i, i.e., there exist a left D-module P_i with $\mathrm{lpd}_D(P_i) = i$ and an injective homomorphism $\epsilon_i \in \hom_D(M, P_i)$.*

Proof. Let us give a constructive proof of the result. Let us first prove the result for a 0-pure module $M = D^{1\times p}/(D^{1\times q}\, R)$, i.e., $t_0(M) = M$ and $t_1(M) = 0$. Since $j_D(M) = 0$, $\ker_D(R.) \cong \hom_D(M, D) \neq 0$ (see Theorem 1), which shows that the Auslander transpose $N_{11} = D^{p_{11}}/(R_{11}\, D^{p_{01}})$ of $M = D^{1\times p_{01}}/(D^{1\times p_{11}}\, R_{11})$ ($R_{11} = R$, $p_{01} = p$, $p_{11} = q$) admits a free resolution of the form $\ldots \xrightarrow{R_{-11}.} D^{p_{-11}} \xrightarrow{R_{01}.} D^{p_{01}} \xrightarrow{R_{11}.} D^{p_{11}} \xrightarrow{\kappa_{11}} N_{11} \longrightarrow 0$, where $R_{01} \neq 0$. Since $T_1 = \mathrm{ext}_D^1(N_{11}, D) \cong M_1 = t_1(M) = 0$ (see Theorem 10), then we get the exact sequence $D^{1\times p_{-11}} \xleftarrow{.R_{01}} D^{1\times p_{01}} \xleftarrow{.R_{11}} D^{1\times p_{11}}$, which yields $M = \mathrm{coker}_D(.R_{11}) \cong \mathrm{im}_D(.R_{01}) \subseteq D^{1\times p_{-11}}$, where $D^{1\times p_{-11}}$ is a free left D-module, i.e., $\mathrm{lpd}(D^{1\times p_{-11}}) = 0$.

Let us now suppose that $i \geq 1$. Since M is i-pure, $j_D(M) = i$. Hence, if (24) is a free resolution of M, then $N_{ii} = D^{p_{ii}}/(R_{ii}\, D^{p_{(i-1)i}})$ admits the free resolution (61), where $R_{ii} = R_i$, $p_{ii} = p_i$, and $p_{i(i+1)} = p_{ii}$ (see the notations of Section 3). Now, $\mathrm{ext}_D^i(M, D) = \ker_D(R_{(i+1)(i+1)}.)/\mathrm{im}_D(R_{ii}.) = (R_{i(i+1)}\, D^{p_{(i-1)(i+1)}})/(R_{ii}\, D^{p_{(i-1)i}})$ is a left D-submodule of the left D-module N_{ii}. Using Proposition 4, we obtain

$$\mathrm{ext}_D^i(M, D) \cong D^{p_{(i-1)(i+1)}}/((F_{(i-1)(i+1)} \quad R_{(i-1)(i+1)})\, D^{p_{(i-1)i}+p_{(i-2)(i+1)}}),$$

and the following commutative exact diagram holds:

$$\begin{array}{ccccccc}
& & & & 0 & & \\
& & & & \downarrow & & \\
D^{p_{(i-1)i}+p_{(i-2)(i+1)}} & \xrightarrow{(F_{(i-1)(i+1)} \quad R_{(i-1)(i+1)}).} & D^{p_{(i-1)(i+1)}} & \longrightarrow & \mathrm{ext}^i_D(M,D) & \longrightarrow & 0 \\
\downarrow \left(I_{p_{(i-1)i}} \quad 0\right). & & \downarrow R_{i(i+1)}. & & \downarrow u & & \\
D^{p_{(i-1)i}} & \xrightarrow{R_{ii}.} & D^{p_{ii}} & \xrightarrow{\kappa_{ii}} & N_{ii} & \longrightarrow & 0.
\end{array}$$

Let $q_0 = p_{(i-1)(i+1)}$, $q_1 = p_{(i-1)i} + p_{(i-2)(i+1)}$, $Q_1 = (F_{(i-1)(i+1)} \quad R_{(i-1)(i+1)})$, $L_0 = R_{i(i+1)}$, and $L_1 = \left(I_{p_{(i-1)i}} \quad 0\right)$. Extending the free resolution of $\mathrm{ext}^i_D(M,D)$, $u \in \hom_D(\mathrm{ext}^i_D(M,D), N_{ii})$ then induces the following commutative exact diagram:

$$\begin{array}{ccccccccccccc}
D^{q_{i+1}} & \xrightarrow{Q_{i+1}.} & D^{q_i} & \xrightarrow{Q_i.} & \dots & \xrightarrow{Q_2.} & D^{q_1} & \xrightarrow{Q_1.} & D^{q_0} & \longrightarrow & \mathrm{ext}^i_D(M,D) & \longrightarrow & 0 \\
\downarrow L_{i+1}. & & \downarrow L_i. & & & & \downarrow L_1. & & \downarrow L_0. & & \downarrow u & & \\
D^{p_{-11}} & \xrightarrow{R_{01}.} & D^{p_{01}} & \xrightarrow{R_{11}.} & \dots & \xrightarrow{R_{(i-1)(i-1)}.} & D^{p_{(i-1)(i-1)}} & \xrightarrow{R_{ii}.} & D^{p_{ii}} & \xrightarrow{\kappa_{ii}} & N_{ii} & \longrightarrow & 0.
\end{array} \tag{91}$$

Since $j_D(M) = i \geq 1$, Theorem 1 shows that $\ker_D(R_{1i}.) \cong \hom_D(M,D) = 0$, i.e., $R_{01} = 0$ (see also Remark 13). Since D is Auslander regular (see Remark 7), $\hom_D(\mathrm{ext}^i_D(M,D),D) = 0$ for $i \geq 1$. Applying the contravariant left exact functor $\hom_D(\,\cdot\,,D)$ to the above commutative exact diagram, we get the following commutative diagram:

$$\begin{array}{ccccccccccc}
D^{1\times q_{i+1}} & \xleftarrow{.Q_{i+1}} & D^{1\times q_i} & \xleftarrow{.Q_i} & \dots & \xleftarrow{.Q_2} & D^{1\times q_1} & \xleftarrow{.Q_1} & D^{1\times q_0} & \longleftarrow & 0 \\
\uparrow & & \uparrow .L_i & & & & \uparrow .L_1 & & \uparrow .L_0 & & \\
0 & \longleftarrow & D^{1\times p_{01}} & \xleftarrow{.R_{11}} & \dots & \xleftarrow{.R_{(i-1)(i-1)}} & D^{1\times p_{(i-1)(i-1)}} & \xleftarrow{.R_{ii}} & D^{1\times p_{ii}} & \longleftarrow & 0.
\end{array} \tag{92}$$

Since D is Auslander regular, $\mathrm{ext}^j_D(\mathrm{ext}^i_D(M,D),D) = 0$ for $j = 1,\dots,i-1$, which shows that the top horizontal complex of (92) is exact at $D^{1\times q_j}$ for $j = 0,\dots,i-1$. The defect of exactness of the top horizontal complex at $D^{1\times q_i}$ is $\mathrm{ext}^i_D(\mathrm{ext}^i_D(M,D),D) \cong \ker_D(.Q_{i+1})/\mathrm{im}_D(.Q_i)$, and the defect of exactness of the bottom horizontal complex at $D^{1\times p_{01}}$ is $\mathrm{ext}^i_D(N_{ii},D) \cong D^{1\times p_{01}}/(D^{1\times p_{11}}\,R_{11}) = M$. Hence, $.L_i$ induces the following canonical left D-homomorphism

$$\begin{array}{rcl}
\varepsilon_i\colon M & \longrightarrow & \ker_D(.Q_{i+1})/\mathrm{im}_D(.Q_i) \cong \mathrm{ext}^i_D(\mathrm{ext}^i_D(M,D),D) \\
\pi(\lambda) & \longmapsto & o(\lambda\, L_i),
\end{array}$$

where $o\colon \ker_D(.Q_{i+1}) \longrightarrow \ker_D(.Q_{i+1})/\mathrm{im}_D(.Q_i)$ is the projection, and $\lambda \in D^{1\times p_{01}}$. Since M is i-pure, 1 of Theorem 8 implies that ε_i is an injective left D-homomorphism.

The exactness of the top horizontal complex of (92) at $D^{1\times q_j}$ for $j = 0,\dots,i-1$ shows that the left D-module $P_i = D^{1\times q_i}/(D^{1\times q_{i-1}}\,Q_i)$ admits a free resolution of length i, which implies that $\mathrm{ext}^j_D(P_i,D) = 0$ for all $j > i$. The free resolution of $\mathrm{ext}^i_D(M,D)$ defined by (92) shows that

$\mathrm{ext}^i_D(P_i, D) \cong \mathrm{ext}^i_D(M, D) \neq 0$, which proves that $\mathrm{lpd}_D(P_i) = i$ by Proposition 2. Finally, since $\ker_D(.Q_{i+1}) \subseteq D^{1\times q_i}$, $\ker_D(.Q_{i+1})/\mathrm{im}_D(.Q_i)$ is a left D-submodule of $P_i = D^{1\times q_i}/(D^{1\times q_{i-1}}\, Q_i)$, ε_i induces an injective left D-homomorphism $\epsilon_i \colon M \longrightarrow P_i$ defined by $\epsilon_i(\pi(\lambda)) = \sigma_i(\lambda\, L_i)$ for all $\lambda \in D^{1\times p_{01}}$, where $\sigma_i : D^{1\times q_i} \longrightarrow P_i$ is the canonical projection onto P_i. □

The constructive proof of Proposition 11 is implemented in the PurityFiltration package.

A proof of Proposition 11 based on Spencer cohomology [51] was recently obtained in [39].

Example 7. Let D be an Auslander regular ring with $\mathrm{gld}(D) = n$ and M a nonzero holonomic left D-module. In particular, $\mathrm{pd}_D(M) \leq n$. By definition of a holonomic module, $j_D(M) = n$, and thus $\mathrm{ext}^n_D(M, D) \neq 0$ and $\mathrm{ext}^i_D(M, D) = 0$ for $i > n$, which proves that $\mathrm{lpd}_D(M) = n$ by Proposition 2. Since M is n-pure, we can take $P_n = M$ and $\epsilon_n = \mathrm{id}_M$ in Proposition 11.

Example 8. Let D be an Auslander regular ring and $M \neq 0$ a left D-module defined by the free resolution $0 \longrightarrow D^{1\times p} \xrightarrow{.R} D^{1\times p} \xrightarrow{\pi} M \longrightarrow 0$. Since $M \cong \mathrm{ext}^1_D(\mathrm{ext}^1_D(M, D), D)$, i.e., M is 1-pure, and $\mathrm{lpd}_D(M) = 1$, we can then take $P_1 = M$ and $\epsilon_1 = \mathrm{id}_M$ in Proposition 11. If D is also a Cohen-Macaulay ring, then $\dim_D(M) = \dim(D) - 1$. If D is the ring of PD operators with coefficients in a differential field K of characteristic 0, then this result proves *Janet's conjecture* [26], which was first obtained by Johnson in [28] (see also [40, 41]).

Corollary 7. *Let D be an Auslander regular ring, $M = D^{1\times p}/(D^{1\times q}\, R)$ an i-pure left D-module, and $\mathcal{F}$ an injective left D-module. Then, there exist two matrices $Q \in D^{s\times r}$ and $L \in D^{p\times r}$ such that the left D-module $P = D^{1\times r}/(D^{1\times s}\, Q)$ is such that $\mathrm{lpd}_D(P) = i$, and*

$$\ker_{\mathcal{F}}(R.) = L\, \ker_{\mathcal{F}}(Q.),$$

i.e., an i-pure linear system is the image of a linear system of projective dimension i.

Proof. The proof of Proposition 11 shows that the following commutative exact diagram holds

$$\begin{tikzcd}
0 & P_i \arrow[l] & D^{1\times q_i} \arrow[l, "\sigma_i"'] & D^{1\times q_{i-1}} \arrow[l, ".Q_i"'] \\
0 & M \arrow[l] \arrow[u, "\epsilon_i"'] & D^{1\times p_{01}} \arrow[l, "\pi"'] \arrow[u, ".L_i"'] & D^{1\times p_{11}}, \arrow[l, ".R_{11}"'] \arrow[u, ".L_{i-1}"'] \\
 & 0 \arrow[u] & &
\end{tikzcd} \tag{93}$$

where $R_{11} = R$, $p_{01} = p$, and $p_{11} = q$. Applying the contravariant exact functor $\hom_D(\,\cdot\,, \mathcal{F})$ to (93), we obtain the following commutative exact diagram

$$\begin{tikzcd}
0 \arrow[r] & \ker_{\mathcal{F}}(Q_i.) \arrow[r] \arrow[d, "\epsilon_i^\star"] & \mathcal{F}^{q_i} \arrow[r, "Q_i."] \arrow[d, "L_i."] & \mathcal{F}^{q_{i-1}} \arrow[d, "L_{i-1}."] \\
0 \arrow[r] & \ker_{\mathcal{F}}(R_{11}.) \arrow[r] & \mathcal{F}^{p_{01}} \arrow[r, "R_{11}."] & \mathcal{F}^{p_{11}},
\end{tikzcd}$$

which shows that $\epsilon_i^\star \colon \ker_{\mathcal{F}}(Q_i.) \longrightarrow \ker_{\mathcal{F}}(R.)$ is defined by $\epsilon_i^\star(\xi) = L_i\, \xi$ for all $\xi \in \ker_{\mathcal{F}}(Q_i.)$. Using Theorem 3, the short exact sequence $0 \longrightarrow M \xrightarrow{\epsilon_i} P_i \longrightarrow \mathrm{coker}\,\epsilon_i \longrightarrow 0$ yields the long exact sequence $0 \longrightarrow \hom_D(\mathrm{coker}\,\epsilon_i, \mathcal{F}) \longrightarrow \hom_D(P_i, \mathcal{F}) \longrightarrow \hom_D(M, \mathcal{F}) \longrightarrow \mathrm{ext}^1_D(\mathrm{coker}\,\epsilon_i, \mathcal{F})$.

Since $\mathcal{F}$ is an injective left D-module, $\mathrm{ext}^1_D(\mathrm{coker}\,\epsilon_i, \mathcal{F}) = 0$ (see Definition 3), which shows that $\epsilon_i^\star$ is surjective, i.e., using Theorem 1, for every $\eta \in \ker_{\mathcal{F}}(R.)$, there exists $\xi \in \ker_{\mathcal{F}}(Q_i.)$ such that $\eta = L_i\,\xi$. We note that $\epsilon_i^\star$ is also injective iff $\hom_D(\mathrm{coker}\,\epsilon_i, \mathcal{F}) \cong \ker_{\mathcal{F}}((L_i^T \quad Q_i)^T.) = 0$. $\square$

Example 9. Let M be the $D = \mathbb{Q}[\partial_1, \partial_2, \partial_3]$-module finitely presented by the following matrix:

$$R = \begin{pmatrix} \partial_1 & 0 \\ 0 & \partial_1 \\ \partial_2 & -\partial_3 \end{pmatrix} \in D^{3\times 2}.$$

Then, the D-module M admits the following free resolution:

$$0 \longleftarrow M \overset{\pi}{\longleftarrow} D^{1\times 2} \overset{.R}{\longleftarrow} D^{1\times 3} \overset{.R_2}{\longleftarrow} D \longleftarrow 0, \quad R_2 = (-\partial_2 \quad \partial_3 \quad \partial_1).$$

Clearly, $\mathrm{ext}^2_D(M, D) = D/(\partial_1, \partial_2, \partial_3) \neq 0$, which shows that $\mathrm{pd}_D(M) = 2$ by Proposition 2. Using Algorithm 1, we can check that $M = M_1 = t(M)$ and $M_2 \cong \mathrm{ext}^2_D(N_{22}, D) = 0$, where $N_{22} = D/(\partial_1, \partial_2, \partial_3)$, which shows that M is a 1-pure D-module. With the notations of Section 3 and of the proof of Proposition 11, i.e., $R_{11} = R$, $R_{22} = R_2$, $\ker_D(R_{22}.) = R_{12}\,D^3$, $\ker_D(R_{12}.) = R_{02}\,D$, $R_{12}\,F_{02} = R_{11}$, $Q_1 = (F_{02} \quad R_{02})$, $L_0 = R_{12}$, and $L_1 = (I_2 \quad 0)$, where

$$R_{12} = \begin{pmatrix} \partial_3 & \partial_1 & 0 \\ \partial_2 & 0 & \partial_1 \\ 0 & \partial_2 & -\partial_3 \end{pmatrix}, \quad F_{02} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad R_{02} = \begin{pmatrix} -\partial_1 \\ \partial_3 \\ \partial_2 \end{pmatrix},$$

we obtain $\mathrm{ext}^1_D(M, D) = \ker_D(R_{22}.)/(R_{11}\,D^2) = (R_{12}\,D^3)/(R_{11}\,D^2) \cong D^3/(Q_1 D^3)$. By Proposition 11, the D-homomorphism $\epsilon \colon M \longrightarrow P_1 = D^{1\times 3}/(D^{1\times 3}\,Q_1)$ defined by $\epsilon_1(\pi(\lambda)) = \sigma_1(\lambda\,L_1)$ is injective. Since the matrix Q_1 has full row rank and $P_1 \neq 0$, $\mathrm{pd}_D(P_1) = 1$, which shows that the 1-pure D-module M can be embedded into the D-module P_1 of projective dimension 1. Finally, if $\mathcal{F} = C^\infty(\mathbb{R}^3)$ is the injective D-module of smooth functions (see Example 2), then

$$\ker_{\mathcal{F}}(Q_1.) = \{(\partial_3\,\phi(x_2, x_3) \quad \partial_2\,\phi(x_2, x_3) \quad -\phi(x_2, x_3))^T \mid \forall\,\phi \in C^\infty(\mathbb{R}^2)\},$$

which gives $\ker_{\mathcal{F}}(R.) = L_1\,\ker_{\mathcal{F}}(Q_1.) = \{(\partial_3\,\phi(x_2, x_3) \quad \partial_2\,\phi(x_2, x_3))^T \mid \forall\,\phi \in C^\infty(\mathbb{R}^2)\}$.

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6 Appendix: The PurityFiltration package

We demonstrate the PurityFiltration package (Maple 15) dedicated to grade filtration and its applications. It uses the OreModules package [15] and the OreMorphisms package [17].

```
> with(OreModules):
> with(OreMorphisms):
> with(PurityFiltration):
```

Since the notation D is protected in Maple, in what follows, we shall use A instead of D.

6.1 Grade filtration of linear PD systems

Example 10. Let A be the ring of PD operators in $d_1 = \frac{\partial}{\partial x_1}$ and $d_2 = \frac{\partial}{\partial x_2}$ with coefficients in $\mathbb{Q}[x_1, x_2]$.

```
> A:=DefineOreAlgebra(diff=[d[1],x[1]],diff=[d[2],x[2]],polynom=[x[1],x[2]]):
```

Let us consider the following matrix $R \in A^{3\times 3}$ of PD operators first considered by Janet and studied in J.-F. Pommaret, "Algebraic analysis of control systems defined by partial differential equations", *Lecture Notes in Control and Inform. Sci.*, 311, Springer, 2005, pp. 155–223.

```
> R:=matrix(3,3,[0,d[2]-d[1],d[2]-d[1],d[2],-d[1],-d[2]-d[1],d[1],-d[1],
> -2*d[1]]);
```

$$R := \begin{bmatrix} 0 & d_2 - d_1 & d_2 - d_1 \\ d_2 & -d_1 & -d_2 - d_1 \\ d_1 & -d_1 & -2\,d_1 \end{bmatrix}$$

Let us compute the grade number $j_A(M)$ of the A-module $M = A^{1\times 3}/(A^{1\times 3}\,R)$.

```
> GradeNumber(R,A);
```

$$0$$

Let us check that $j_A(M) = \mathrm{codim}_A(M)$ by computing the codimension of M.

```
> Codimension(R,A);
```

$$0$$

Let us check whether or not M is a pure A-module.

```
> IsPure(R,A);
```

$$\mathit{false}$$

Since M is not a pure A-module, it admits a nontrivial grade filtration. Let us compute it.

```
> G:=GradeFiltrationByGenerators(R,A);
```

$$G := [[\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}, 1], [[\begin{array}{ccc} -1 & 1 & 2 \end{array}], 2]]$$

We obtain that the A-modules $M_1 = (A^{1\times 2}\,G_{11})/(A^{1\times 3}\,R)$ and $M_2 = (A\,G_{21})/(A^{1\times 3}\,R)$ define the grade filtration of M, where G_{i1} is the first matrix of the i^{th} entry of G (the second entry G_{i2} is the index i of the submodule M_i). If $\pi\colon A^{1\times 3} \longrightarrow M$ is the canonical projection onto M, $\{f_j\}_{j=1,2,3}$ the standard basis of $A^{1\times 3}$, $\{y_j = \pi(f_j)\}_{j=1,2,3}$ a family of generators of M, and $y = (y_1 \quad y_2 \quad y_3)^T$, then M is defined by the relations $R\,y = 0$. Then, we have:

$$\begin{cases} M_0 = M = A\,y_1 + A\,y_2 + A\,y_3, \\ M_1 = A\,(y_1 - y_3) + A\,(y_2 + y_3), \\ M_2 = A\,(-y_1 + y_1 + 2\,y_3), \\ M_3 = 0. \end{cases}$$

If an option is added to the command `GradeFiltrationByGenerators`, then we can also obtain the PD equations satisfied by the generators of the A-module M_i for $i = 0, 1, 2$. The PD operators annihilating the j^{th} generators of M_i are the entries of j^{th} block-diagonal matrix of the matrix in front of the matrix G_{i1}, i.e.,

```
> H:=GradeFiltrationByGenerators(R,A,opt);
```

$$H := [[\begin{bmatrix} -d_2+d_1 & 0 \\ 0 & -d_2+d_1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}, 1], [\begin{bmatrix} d_2 \\ d_1 \end{bmatrix}, \begin{bmatrix} -1 & 1 & 2 \end{bmatrix}, 2]]$$

shows that $z_1 = y_1 - y_3$ (resp., $z_2 = y_2 + y_3$) satisfies the PD operators appearing in the first (resp., second) block-diagonal matrix of the matrix appearing in front of G_{i1}, i.e., $(d_1 - d_2)\, z_1 = 0$ (resp., $(d_1 - d_2)\, z_2 = 0$). The generator $z_3 = -y_1 + y_2 + 2\, y_3$ of M_2 satisfies $d_2\, z_3 = 0$ and $d_1\, z_3 = 0$.

A presentation matrix of the A-module M_i/M_{i+1} is computed by the command `PureFactors_NR`:

```
> J:=PureFactors_NR(R,A);
```

$$J := [\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & -d_2+d_1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & d_1 \\ 0 & d_2 \end{bmatrix}]$$

We get $M/M_1 = A^{1\times 3}/(A^{1\times 2}\, J_1)$, $M_1/M_2 = A^{1\times 2}/(A^{1\times 2}\, J_2)$, and $M_2 = A^{1\times 2}/(A^{1\times 3}\, J_3)$, where J_i is the i^{th} matrix of J. The suffix `_NR` stands for "NonReduced", i.e., the matrix J_i's does not generally form a Gröbner basis or is not simplified. To obtain such a presentation matrix of the A-module M_i/M_{i+1} for $i = 0, 1, 2$, we can use the command `PureFactors`

```
> F:=PureFactors(R,A);
```

$$F := [\begin{bmatrix} 0 \end{bmatrix}, \begin{bmatrix} -d_2+d_1 \end{bmatrix}, \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}]$$

i.e., we have:

$$\begin{cases} M/M_1 \cong A/(A\, F_1) \cong A, \\ M_1/M_2 \cong A/(A\, F_2) = A/(A\,(d_1 - d_2)), \\ M_2 \cong A/(A^{1\times 2}\, F_3) = A/(A\, d_1 + A\, d_2). \end{cases}$$

Let us compute the codimension of the A-module M_i/M_{i+1} for $i = 0, 1, 2$:

```
> map(Codimension,F,A);
```

$$[0, 1, 2]$$

Thus, $\operatorname{codim}_A(M/M_1) = 0$, $\operatorname{codim}_A(M_1/M_2) = 1$, and $\operatorname{codim}_A(M_2) = 2$, i.e., $\dim_A(M/M_1) = 2$, $\dim_A(M_1/M_2) = 1$, and $\dim_A(M_2) = 0$.

Let us now check that the A-module M_i/M_{i+1} is i-pure for $i = 0, 1, 2$:

```
> map(IsPure,F,A);
```

$$[0, 1, 2]$$

Another way to define the grade filtration $\{M_i\}_{i=0,\ldots,2}$ of M is by means of finitely presented A-modules $L_i \cong M_i$ and injective $\theta_i \in \hom_A(L_i, M)$ for $i = 1, 2$ (see Algorithm 3).

```
> H:=GradeFiltrationByMorphisms(R,A);
```

$$H := [[\begin{bmatrix} 0 & -d_2+d_1 \\ d_2 & -d_2 \\ d_1 & -d_2 \end{bmatrix}, \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}], [\begin{bmatrix} 1 & 0 \\ 0 & d_1 \\ 0 & d_2 \end{bmatrix}, \begin{bmatrix} 0 & d_2-d_1 & d_2-d_1 \\ -1 & 1 & 2 \end{bmatrix}]]$$

We have $L_1 = A^{1\times 2}/(A^{1\times 3}\, H_{11})$ and $L_2 = A^{1\times 2}/(A^{1\times 3}\, H_{21})$, where H_{i1} is the first matrix in the i^{th} entry of H. Moreover, the injective A-homomorphism $\theta_i\colon L_i \longrightarrow M$ is defined by $\theta_i(\rho'_i(\lambda)) = \pi(\lambda\, H_{i2})$, where H_{i2} is the second matrix in the i^{th} entry of H and ρ'_i is the canonical projection onto L_i. Let us check again that the A-homomorphisms θ_i's are injective.

```
> seq(TestInj(H[i][1],R,H[i][2],A),i=1..2);
```

$$\mathit{true}, \mathit{true}$$

Let us now compute an A-module $\overline{M}$ isomorphic to M which is finitely presented by the matrix $\overline{R}$ defined by means of the grade filtration of M (see Theorem 12).

```
> P:=PurePresentation_NR(R,A);
```

$$P := [\begin{bmatrix} 0 & d_2-d_1 & d_2-d_1 \\ d_2 & -d_1 & -d_2-d_1 \\ d_1 & -d_1 & -2\,d_1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & d_2-d_1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -d_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & -d_1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & d_2-d_1 & d_2-d_1 \\ -1 & 1 & 2 \end{bmatrix}]$$

We get $\overline{M} = A^{1\times 7}/(A^{1\times 7}\, P_2) \cong M = A^{1\times 3}/(A^{1\times 3}\, P_1)$, where P_i is the i^{th} matrix of P. If $\overline{\pi}$ is the canonical projection onto $\overline{M}$, then $\varphi\colon M \longrightarrow \overline{M}$ defined by $\varphi(\pi(\lambda)) = \overline{\pi}(\lambda\, P_3)$ is an isomorphism, whose inverse $\varphi^{-1}\colon \overline{M} \longrightarrow M$ is $\varphi^{-1}(\overline{\pi}(\mu)) = \pi(\mu\, P_4)$.

Let us check that φ is an isomorphism and φ^{-1} is defined by P_4.

```
> TestIso(P[1],P[2],P[3],A);
```

$$\mathit{true}$$

```
> TestIso(P[2],P[1],P[4],A);
```

$$\mathit{true}$$

The matrix $\overline{R}$, defined by the above matrices J_i's, can be simplified by computing a Gröbner basis of the A-module defined by the matrix J_i for $i = 1, 2, 3$. This can be obtained by using the command `PurePresentation`:

```
> Q:=PurePresentation(R,A);
```

$$Q := \left[\begin{bmatrix} 0 & d_2-d_1 & d_2-d_1 \\ d_2 & -d_1 & -d_2-d_1 \\ d_1 & -d_1 & -2\,d_1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & -d_2+d_1 & 0 \\ 0 & 0 & d_1 \\ 0 & 0 & d_2 \end{bmatrix}, \begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & 1 & 2 \end{bmatrix} \right]$$

We obtain $M = A^{1\times 3}/(A^{1\times 3}\,Q_1) \cong L = A^{1\times 3}/(A^{1\times 3}\,Q_2)$, where Q_i is the i^{th} matrix of Q. The isomorphism $\psi\colon M \longrightarrow L$ is defined by $\psi(\pi(\lambda)) = \vartheta(\lambda\, Q_3)$, where ϑ is the canonical projection onto L. Let us check that ψ is an isomorphism.

```
> TestIso(Q[1],Q[2],Q[3],A);
```

$$true$$

Now, $\psi^{-1}\colon L \longrightarrow M$ is defined by $\psi^{-1}(\vartheta(\mu)) = \pi(\mu\, Q_4)$.

```
> TestIso(Q[2],Q[1],Q[4],A);
```

$$true$$

The presentation matrix Q_2 of the A-module L is defined by the presentation matrices F_i's of the pure A-modules M_i/M_{i+1} for $i = 0, 1, 2$. The fact that $F_1 = 0$ explains why the first row of Q_2 is 0. The presentation matrix Q_2 can be again simplified using the command `SimplifiedPresentation`.

```
> S:=SimplifiedPresentation(Q[2],A);
```

$$S := \left[\begin{bmatrix} 0 & -d_2+d_1 & 0 \\ 0 & 0 & d_1 \\ 0 & 0 & d_2 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \right]$$

We have $L = A^{1\times 3}/(A^{1\times 3}\,S_1)$, where S_1 is the first matrix of S (the second and the third matrices S_2 and S_3 defining the identity homomorphism between the two different presentations of L).

Let us compute a presentation of the A-module $M_1 = t(M)$ based on the terms $\{M_i\}_{i=1,2}$ of the grade filtration of M_1.

```
> T:=PurePresentationOfTorsionSubmodule(R,A);
```

$$T := \left[\begin{bmatrix} d_2-d_1 & 0 \\ 0 & d_1 \\ 0 & d_2 \end{bmatrix}, \begin{bmatrix} 0 & d_2-d_1 & d_2-d_1 \\ d_2 & -d_1 & -d_2-d_1 \\ d_1 & -d_1 & -2\,d_1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \\ -1 & 1 & 2 \end{bmatrix} \right]$$

The first (resp., second) matrix T_1 (resp., T_2) of T is a presentation of $t(M)$ (resp., M), i.e., $t(M) \cong K = A^{1\times 2}/(A^{1\times 3}\,T_1)$ (resp., $M = A^{1\times 3}/(A^{1\times 3}\,T_2)$). The third matrix T_3 of T defines the embedding of the A-module K into M, i.e., defines an injective $\iota \in \hom_A(K, M)$ defined by $\iota(\sigma(\nu)) = \pi(\nu\, T_3)$, where $\sigma\colon A^{1\times 2} \longrightarrow K$ is the canonical projection onto K.

```
> TestInj(T[1],T[2],T[3],A);
```

$$true$$

The form of the matrix S_1 shows that $L \cong A \oplus K$, and the form of the matrix T_1 shows that $t(M) = M_1 = M_1/M_2 \oplus M_2$. Thus, we obtain:

$$M = A \oplus M_1/M_2 \oplus M_2 = A \oplus A/(A\,(d_2 - d_1)) \oplus A/(A\,d_1 + A\,d_2).$$

Let us finally check that K is a torsion A-module, i.e., $\mathrm{codim}_A(M) \geq 1$.

```
> Codimension(T[1],A);
```

$$1$$

Example 11. Let A be the ring of PD operators in $d_1 = \frac{\partial}{\partial x_1}$, $d_2 = \frac{\partial}{\partial x_2}$, and $d_3 = \frac{\partial}{\partial x_3}$ with coefficients in $\mathbb{Q}[x_1, x_2, x_3]$

```
> A:=DefineOreAlgebra(diff=[d[1],x[1]],diff=[d[2],x[2]],diff=[d[3],x[3]],
> polynom=[x[1],x[2],x[3]]):
```

and R the system matrix of the linear PD system defined by the left hand side of (90):

```
> R:=matrix(6,4,[0,-2*d[1],d[3]-2*d[2]-d[1],-1,0,d[3]-2*d[1],2*d[2]-3*d[1],
> 1,d[3],-6*d[1],-2*d[2]-5*d[1],-1,0,d[2]-d[1],d[2]-d[1],0,d[2],-d[1],
> -d[2]-d[1],0,d[1],-d[1],-2*d[1],0]);
```

$$R := \begin{bmatrix} 0 & -2\,d_1 & d_3 - 2\,d_2 - d_1 & -1 \\ 0 & d_3 - 2\,d_1 & 2\,d_2 - 3\,d_1 & 1 \\ d_3 & -6\,d_1 & -2\,d_2 - 5\,d_1 & -1 \\ 0 & d_2 - d_1 & d_2 - d_1 & 0 \\ d_2 & -d_1 & -d_2 - d_1 & 0 \\ d_1 & -d_1 & -2\,d_1 & 0 \end{bmatrix}$$

Let us study the A-module $M = A^{1\times 4}/(A^{1\times 6}\,R)$. Let us first compute its grade number $j_A(M)$.

```
> GradeNumber(R,A);
```

$$0$$

Let us check that $j_A(M) = \mathrm{codim}_A(M)$ by computing the codimension of M.

```
> Codimension(R,A);
```

$$0$$

Let us check whether or not M is a pure A-module.

```
> IsPure(R,A);
```

false

Let us now compute the grade filtration $\{M\}_{i=0,\dots,3}$ of M:

```
> G:=GradeFiltrationByGenerators(R,A);
```

$$G := [[\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -2\,d_2 + d_3 + d_1 & -1 \end{bmatrix}, 1], [\begin{bmatrix} 0 & 1 & 1 & 0 \\ -1 & 1 & 2 & 0 \end{bmatrix}, 2], [[\begin{array}{cccc} 1 & -1 & -2 & 0 \end{array}], 3]]$$

We have $0 \subseteq M_3 \subseteq M_2 \subseteq M_1 \subseteq M_0 = M$, where

$$\begin{cases} M_0 = A^{1\times 4}/(A^{1\times 6}\, R), \\ M_1 = (A^{1\times 3}\, G_{11})/(A^{1\times 6}\, R), \\ M_2 = (A^{1\times 2}\, G_{21})/(A^{1\times 6}\, R), \\ M_3 = (A\, G_{31})/(A^{1\times 6}\, R), \\ M_4 = 0, \end{cases}$$

where G_{i1} is the first matrix of the i^{th} entry of G (the second entry G_{i2} is the index i of the submodule M_i). Equivalently, if $\pi\colon A^{1\times 4} \longrightarrow M$ is the canonical projection, $\{f_j\}_{j=1,\ldots,4}$ the standard basis of $A^{1\times 4}$, and $\{y_j = \pi(f_j)\}_{j=1,\ldots,4}$ a family of generators of M, then:

$$\begin{cases} M_0 = A\, y_1 + A\, y_2 + A\, y_3 + A\, y_4, \\ M_1 = A\, (y_1 - y_3) + A\, (y_2 + y_3) + A\, ((-2\, d_2 + d_3 + d_1)\, y_3 - y_4), \\ M_2 = A\, (y_2 + y_3) + A\, (-y_1 + y_2 + 2\, y_3), \\ M_3 = A\, (y_1 - y_2 - 2\, y_3), \\ M_4 = 0. \end{cases}$$

If we add an option to the command `GradeFiltrationByGenerators`, then we also obtain the annihilators of the above family of generators of the A-modules M_i's (see Algorithm 2):

```
>   GradeFiltrationByGenerators(R,A,opt);
```

$$\left[\left[\begin{array}{ccc} 4\, d_2 - d_3 & 0 & 0 \\ 4\, d_1 - d_3 & 0 & 0 \\ 0 & 4\, d_2 - d_3 & 0 \\ 0 & 4\, d_1 - d_3 & 0 \\ 0 & 0 & 4\, d_2 - d_3 \\ 0 & 0 & 4\, d_1 - d_3 \end{array} \right], \left[\begin{array}{cccc} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -2\, d_2 + d_3 + d_1 & -1 \end{array} \right], 1 \right],$$

$$\left[\left[\begin{array}{cc} 4\, d_2 - d_3 & 0 \\ 4\, d_1 - d_3 & 0 \\ 0 & d_3 \\ 0 & d_2 \\ 0 & d_1 \end{array} \right], \left[\begin{array}{cccc} 0 & 1 & 1 & 0 \\ -1 & 1 & 2 & 0 \end{array} \right], 2 \right], \left[\left[\begin{array}{c} d_3 \\ d_2 \\ d_1 \end{array} \right], \left[\begin{array}{cccc} 1 & -1 & -2 & 0 \end{array} \right], 3 \right] \right]$$

The matrix in front of G_{i1} defines the PD operators which annihilate the generators of M_i (which are defined by the residue class of the rows of G_i in M). For instance, the first generator $z_1 = y_1 - y_3$ of M_1 satisfies $(4\, d_2 - d_3)\, z_1 = 0$ and $(4\, d_1 - d_3)\, z_1 = 0$ (similarly for the second $z_2 = y_2 + y_3$ and third generator $z_3 = (-2\, d_2 + d_3 + d_1)\, y_3 - y_4$ of M_1). Similarly, M_2 is generated by z_2 and $z_3 = -y_1 + y_2 + 2\, y_3$ which satisfies $d_i\, z_3 = 0$ for $i = 1, 2, 3$. Finally, z_3 generates M_3 and satisfies $d_i\, z_3 = 0$ for $i = 1, 2, 3$.

Another way to define the grade filtration $\{M_i\}_{i=0,\ldots,3}$ of M is by means of finitely presented A-modules $L_i \cong M_i$ and injective $\theta_i \in \hom_A(L_i, M)$ for $i = 1, 2, 3$ (see Algorithm 3).

```
>   H:=GradeFiltrationByMorphisms(R,A);
```

$$H := [[\begin{bmatrix} 0 & 2\,d_1 & -1 \\ 0 & 2\,d_2 & -1 \\ d_3 & 0 & -2 \\ 2\,d_2 & 0 & -1 \\ 2\,d_1 & 0 & -1 \\ 0 & d_3 & -2 \\ 0 & 0 & -d_3+4\,d_2 \\ 0 & 0 & 4\,d_1-d_3 \end{bmatrix}, \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -2\,d_2+d_3+d_1 & -1 \end{bmatrix}],$$

$$[\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & d_1 \\ 0 & 0 & d_2 \\ 0 & 0 & d_3 \\ 0 & 4\,d_1-d_3 & 0 \\ 0 & -d_3+4\,d_2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -2\,d_1 & d_3-2\,d_2-d_1 & -1 \\ 0 & 1 & 1 & 0 \\ -1 & 1 & 2 & 0 \end{bmatrix}], [\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}, \begin{bmatrix} 1 & -1 & -2 & 0 \end{bmatrix}]]$$

We have $L_1 = A^{1\times 3}/(A^{1\times 8}\,H_{11})$, $L_2 = A^{1\times 3}/(A^{1\times 6}\,H_{21})$, and $L_3 = A/(A^{1\times 3}\,H_{31})$, where H_{i1} is the first matrix in the i^{th} entry of H. Moreover, the injective A-homomorphism $\theta_i \colon L_i \longrightarrow M$ is defined by $\theta_i(\rho_i'(\lambda)) = \pi(\lambda\,H_{i2})$, where H_{i2} is the second matrix in the i^{th} entry of H and ρ_i' is the canonical projection onto L_i. Let us check again that the A-homomorphisms θ_i's are injective.

```
> seq(TestInj(H[i][1],R,H[i][2],A),i=1..3);
```

$$true, true, true$$

Let us now compute a presentation of the pure A-modules M_i/M_{i+1} for $i = 0, \ldots, 3$:

```
> J:=PureFactors_NR(R,A);
```

$$J := [\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -2\,d_2+d_3+d_1 & -1 \end{bmatrix}, \begin{bmatrix} 0 & -2\,d_1 & 1 \\ 0 & 1 & 0 \\ -1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ -1 & -4\,d_1+d_3 & 0 \\ 1 & -4\,d_1+d_3 & -d_3 \\ 0 & d_2-d_1 & 0 \\ 0 & d_2-d_1 & -d_2 \\ 0 & 0 & -d_1 \end{bmatrix}, \begin{bmatrix} d_2 \\ d_3 \\ 0 \\ -d_1 \end{bmatrix}]$$

If J_i is the i^{th} matrix of J, then $M/M_1 = A^{1\times 4}/(A^{1\times 3}\,J_1)$, $M_1/M_2 = A^{1\times 3}/(A^{1\times 3}\,J_2)$, $M_2/M_3 = A^{1\times 3}/(A^{1\times 7}\,J_3)$, and $M_3 = A/(A^{1\times 4}\,J_4)$.

Let us compute the codimension of the A-module M_i/M_{i+1} for $i = 0, \ldots, 2$:

```
> map(Codimension,J,A);
```

$$[0, \infty, 2, 3]$$

In particular, we have $\mathrm{codim}_A(M_1/M_2)=\infty$, i.e., $M_1=M_2$. Let us now check that that the A-module M_i/M_{i+1} is either 0 or i-pure for $i=0,\ldots,3$:

```
> map(IsPure,F,A);
```

$$[0,\infty,2,3]$$

The presentation matrix J_i of M_i/M_{i+1} does not generally form a Gröbner basis or is not simplified, which explains the suffix `NR` of the command `PureFactors_NR`, which stands for "NonReduced". To get such a presentation, we can use the command `PureFactors_R`, where R stands for "Reduced":

```
> K:=PureFactors_R(R,A);
```

$$K:=\left[\begin{bmatrix}1&0&-1&0\\0&1&1&0\\0&0&-2\,d_2+d_3+d_1&-1\end{bmatrix},\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix},\begin{bmatrix}1&0&0\\0&0&1\\0&4\,d_1-d_3&0\\0&-d_3+4\,d_2&0\end{bmatrix},\begin{bmatrix}d_1\\d_2\\d_3\end{bmatrix}\right]$$

Hence, $M/M_1=A^{1\times 4}/(A^{1\times 3}\,K_1)$, $M_1/M_2=A^{1\times 3}/(A^{1\times 3}\,K_2)$, $M_2/M_3=A^{1\times 3}/(A^{1\times 4}\,K_3)$, and $M_3=A/(A^{1\times 3}\,K_4)$, where K_i is the i^{th} matrix of K.

We can simplify again the presentation of the A-module M_i/M_{i+1} for $i=0,\ldots,3$ by means of the elementary operations. This can be obtained by the command `PureFactors`.

```
> F:=PureFactors(R,A);
```

$$F:=\left[\begin{bmatrix}0\end{bmatrix},\begin{bmatrix}1\end{bmatrix},\begin{bmatrix}4\,d_1-d_3\\4\,d_2-d_3\end{bmatrix},\begin{bmatrix}d_1\\d_2\\d_3\end{bmatrix}\right]$$

If F_i is the i^{th} matrix of F, then $M/M_1\cong A/(A\,F_1)=A$, $M_1/M_2\cong A/(A\,F_2)=A/A=0$, $M_2/M_3\cong A/(A^{1\times 2}\,F_3)$, and $M_3=A/(A^{1\times 3}\,F_4)$.

Let us check whether or not the A-module M_i/M_{i+1} is 0 or i-pure for $i=0,\ldots,3$.

```
> map(IsPure,K,A);
```

$$[0,\infty,2,3]$$

Let us compute a finite presentation of the A-module M based on the presentation of the pure factors $M_i/M_{i+1}=\mathrm{coker}_A(.F_i)$ for $i=0,\ldots,3$.

```
> P:=PurePresentation_NR(R,A):
```

We get that the A-module M finitely presented by the matrix P_1 defined by

```
> P[1];
```

$$\begin{bmatrix}0&-2\,d_1&d_3-2\,d_2-d_1&-1\\0&d_3-2\,d_1&2\,d_2-3\,d_1&1\\d_3&-6\,d_1&-2\,d_2-5\,d_1&-1\\0&d_2-d_1&d_2-d_1&0\\d_2&-d_1&-d_2-d_1&0\\d_1&-d_1&-2\,d_1&0\end{bmatrix}$$

is isomorphic to the A-module $\overline{M}$ finitely presented by the matrix P_2 defined by:

```
> P[2];
```

$$\begin{bmatrix} 1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2\,d_2+d_3+d_1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2\,d_1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -4\,d_1+d_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -4\,d_1+d_3 & -d_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_2-d_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_2-d_1 & -d_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -d_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -d_1 \end{bmatrix}$$

In other words, $M = A^{1\times 4}/(A^{1\times 6}\,P_1) \cong \overline{M} = A^{1\times 11}/(A^{1\times 17}\,P_2)$ and P_2 is the block-triangular matrix defined in Theorem 12. The corresponding isomorphism is defined by the following matrix

```
> P[3];
```

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

i.e., $\varphi : M \longrightarrow \overline{M}$ is defined by $\varphi(\pi(\lambda)) = \overline{\pi}(\lambda\,P_3)$, where $\overline{\pi} : A^{1\times 11} \longrightarrow \overline{M}$ is the canonical projection onto $\overline{M}$. Let us check again that φ is an isomorphism.

```
> TestIso(R,P[2],P[3],A);
```

$$\mathit{true}$$

Moreover, φ^{-1} is defined by $\varphi^{-1}(\overline{\pi}(\mu)) = \pi(\mu\,P_4)$, where P_4 is defined by:

```
> P[4];
```

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -2\,d_2 + d_3 + d_1 & -1 \\ 0 & -2\,d_1 & d_3 - 2\,d_2 - d_1 & -1 \\ 0 & 1 & 1 & 0 \\ -1 & 1 & 2 & 0 \\ 1 & -1 & -2 & 0 \end{bmatrix}$$

Let us check again that the A-homomorphism from $\overline{M}$ to M defined by P_4 is an isomorphism.

```
>  TestIso(P[2],R,P[4],A);
```

$$true$$

Let us now compute another presentation matrix Q of the A-module M whose diagonal blocks are the presentation matrices K_i's of the pure A-modules M_i/M_{i+1}'s.

```
>  Q:=PurePresentation_R(R,A):
```

We get that the A-module M finitely presented by the matrix Q_1 defined by

```
>  Q[1];
```

$$\begin{bmatrix} 0 & -2\,d_1 & d_3 - 2\,d_2 - d_1 & -1 \\ 0 & d_3 - 2\,d_1 & 2\,d_2 - 3\,d_1 & 1 \\ d_3 & -6\,d_1 & -2\,d_2 - 5\,d_1 & -1 \\ 0 & d_2 - d_1 & d_2 - d_1 & 0 \\ d_2 & -d_1 & -d_2 - d_1 & 0 \\ d_1 & -d_1 & -2\,d_1 & 0 \end{bmatrix}$$

is isomorphic to the A-module $\overline{\overline{M}}$ finitely presented by the matrix Q_2 defined by

```
>  Q[2];
```

$$\left[\begin{array}{ccccccccccc}
1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -2\,d_2+d_3+d_1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -2\,d_1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4\,d_1-d_3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -d_3+4\,d_2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_3
\end{array}\right]$$

i.e., $M = A^{1\times 4}/(A^{1\times 6}\,Q_1) \cong \overline{\overline{M}} = A^{1\times 11}/(A^{1\times 13}\,Q_2)$. The isomorphism $\psi\colon M \longrightarrow \overline{\overline{M}}$ is defined by the following matrix

```
>  Q[3];
```

$$\left[\begin{array}{ccccccccccc}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]$$

i.e., $\psi(\pi(\lambda)) = \overline{\overline{\pi}}(\lambda\,Q_3)$. Let us check again that ψ is an isomorphism.

```
>  TestIso(R,Q[2],Q[3],A);
```

$$\mathit{true}$$

Moreover, $\psi^{-1}\colon \overline{\overline{M}} \longrightarrow M$ is defined by $\psi^{-1}(\overline{\overline{\pi}}(\mu)) = \pi(\mu\,Q_4)$, where Q_4 is defined by:

```
>  Q[4];
```

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -2\,d_2+d_3+d_1 & -1 \\ 0 & -2\,d_1 & d_3-2\,d_2-d_1 & -1 \\ 0 & 1 & 1 & 0 \\ -1 & 1 & 2 & 0 \\ 1 & -1 & -2 & 0 \end{bmatrix}$$

Let us check again that ψ^{-1} is an isomorphism.

```
>   TestIso(Q[2],R,Q[4],A);
```

true

We can simplify again the presentation matrix Q_2 by means of elementary operations. This can be achieved using the command `PurePresentation`.

```
>   S:=PurePresentation(R,A);
```

$$S := \left[\begin{bmatrix} 0 & -2\,d_1 & d_3-2\,d_2-d_1 & -1 \\ 0 & d_3-2\,d_1 & 2\,d_2-3\,d_1 & 1 \\ d_3 & -6\,d_1 & -2\,d_2-5\,d_1 & -1 \\ 0 & d_2-d_1 & d_2-d_1 & 0 \\ d_2 & -d_1 & -d_2-d_1 & 0 \\ d_1 & -d_1 & -2\,d_1 & 0 \end{bmatrix}, \begin{bmatrix} -4\,d_1+d_3 & 0 & 0 \\ -4\,d_2+d_3 & 0 & 0 \\ 0 & 0 & d_1 \\ 0 & 0 & d_2 \\ 0 & 0 & d_3 \end{bmatrix}, \right.$$
$$\left. \begin{bmatrix} -1 & -1 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & 0 \\ 2\,d_1 & 2\,d_2-d_1-d_3 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & -1 & -2 & 0 \end{bmatrix} \right]$$

We obtain $M \cong L = A^{1\times 3}/(A^{1\times 5}\,S_2)$, where S_2 is the second matrix of S. The isomorphism $\varphi\colon M \longrightarrow L$ is defined by $\varphi(\pi(\lambda)) = \vartheta(\lambda\,S_3)$, where S_3 is the third matrix of S, $\lambda \in A^{1\times 4}$, and $\vartheta\colon A^{1\times 3} \longrightarrow L$ is the canonical projection onto L.

Let us check again that φ is an isomorphism.

```
>   TestIso(R,S[2],S[3],A);
```

true

Moreover, $\varphi^{-1}\colon L \longrightarrow M$ is defined $\varphi^{-1}(\vartheta(\mu)) = \pi(\mu\,S_4)$ for all $\mu \in A^{1\times 3}$, where S_4 is the fourth matrix of S.

```
> TestIso(S[2],R,S[4],A);
```

$$true$$

From the presentation matrix S, we get $M \cong M_3 \oplus M_1/M_3 \oplus A$.

A presentation of the torsion submodule $t(M) = M_1$ of M based on the terms $\{M_i\}_{i=1,2,3}$ of the grade filtration of M_1 can be computed using the command `PurePresentationOfTorsionSubmodule`.

```
> T:=PurePresentationOfTorsionSubmodule(R,A);
```

$$T := \left[\begin{bmatrix} 4\,d_1 - d_3 & 0 \\ 4\,d_2 - d_3 & 0 \\ 0 & d_1 \\ 0 & d_2 \\ 0 & d_3 \end{bmatrix}, \begin{bmatrix} 0 & -2\,d_1 & d_3 - 2\,d_2 - d_1 & -1 \\ 0 & d_3 - 2\,d_1 & 2\,d_2 - 3\,d_1 & 1 \\ d_3 & -6\,d_1 & -2\,d_2 - 5\,d_1 & -1 \\ 0 & d_2 - d_1 & d_2 - d_1 & 0 \\ d_2 & -d_1 & -d_2 - d_1 & 0 \\ d_1 & -d_1 & -2\,d_1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & -1 & -2 & 0 \end{bmatrix} \right]$$

We can check that the A-homomorphism $\iota \colon t(M) = A^{1\times 2}/(A^{1\times 5}\,T_1) \longrightarrow M = A^{1\times 4}/(A^{1\times 6}\,T_2)$ defined by $\iota(\sigma(\nu)) = \pi(\nu\,T_3)$ is injective.

```
> TestInj(T[1],T[2],T[3],A);
```

$$true$$

Let us check that the A-module finitely presented by T_1 is torsion.

```
> Codimension(T[1],A);
```

$$2$$

Let us compute a solution of the linear system $\ker_{\mathcal{F}}(T_1.) \cong \hom_D(t(M), \mathcal{F})$.

```
> z:=IntegrationOfTorsionDSubmodule(R,A);
```

$$z := \begin{bmatrix} \_F1\ (1/4\,x_2 + 1/4\,x_1 + x_3) \\ \_C1 \end{bmatrix}$$

Let us check that z is a solution of $\ker_{\mathcal{F}}(T_1.)$.

```
> ApplyMatrix(T[1],z,A);
```

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Finally, let us try to integrate the linear system $\ker_{\mathcal{F}}(R.) \cong \hom_D(M, \mathcal{F})$. We obtain

```
> y:=IntegrationOfDModule(R,A,xi);
```

$$y := \begin{bmatrix} \_y_3\,(x_1,x_2,x_3) + \_\mathit{F1}\,(1/4\,x_2 + 1/4\,x_1 + x_3) + \_\mathit{C1} - \xi_1\,(x_1,x_2,x_3) \\ -\_y_3\,(x_1,x_2,x_3) + \_\mathit{F1}\,(1/4\,x_2 + 1/4\,x_1 + x_3) + \xi_1\,(x_1,x_2,x_3) \\ \_y_3\,(x_1,x_2,x_3) - \xi_1\,(x_1,x_2,x_3) \\ -1/2\,\mathrm{D}\,(\_\mathit{F1})\,(1/4\,x_2 + 1/4\,x_1 + x_3) + \frac{\partial}{\partial x_1}\_y_3\,(x_1,x_2,x_3) - 2\,\frac{\partial}{\partial x_2}\_y_3\,(x_1,x_2,x_3) \\ +\frac{\partial}{\partial x_3}\_y_3\,(x_1,x_2,x_3) - \frac{\partial}{\partial x_1}\xi_1\,(x_1,x_2,x_3) + 2\,\frac{\partial}{\partial x_2}\xi_1\,(x_1,x_2,x_3) - \frac{\partial}{\partial x_3}\xi_1\,(x_1,x_2,x_3) \end{bmatrix}$$

where ξ and y_3 are two arbitrary functions of x_1, x_2, x_3 (their difference can be replaced in y by a single function of x_1, x_2, x_3), F_1 an arbitrary function of 1 variable, and $\_C_1$ an arbitrary constant. Let us finally check that y is a solution of $\ker_{\mathcal{F}}(R.)$.

```
>  ApplyMatrix(R,y,A);
```

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Example 12. Let A be the ring of PD operators in $d_1 = \frac{\partial}{\partial x_1}$, $d_2 = \frac{\partial}{\partial x_2}$, $d_3 = \frac{\partial}{\partial x_3}$, and $d_4 = \frac{\partial}{\partial x_4}$ with coefficients in $\mathbb{Q}[x_1, x_2, x_3, x_4]$

```
>  A:=DefineOreAlgebra(diff=[d[1],x[1]],diff=[d[2],x[2]],diff=[d[3],x[3]],
>  diff=[d[4],x[4]],polynom=[x[1],x[2],x[3],x[4]]):
```

and the linearized Einstein equations in the vacuum defined by the following matrix $R \in A^{10\times 10}$:

```
>  R := evalm(
>  [[d[2]^2+d[3]^2-d[4]^2, d[1]^2, d[1]^2, -d[1]^2, -2*d[1]*d[2], 0, 0,
>  -2*d[1]*d[3], 0, 2*d[1]*d[4]],
>  [d[2]^2, d[1]^2+d[3]^2-d[4]^2, d[2]^2, -d[2]^2, -2*d[1]*d[2], -2*d[2]*d[3],
>  0, 0, 2*d[2]*d[4], 0],
>  [d[3]^2, d[3]^2, d[1]^2+d[2]^2-d[4]^2, -d[3]^2, 0, -2*d[2]*d[3], 2*d[3]*d[4],
>  -2*d[1]*d[3], 0, 0],
>  [d[4]^2, d[4]^2, d[4]^2, d[1]^2+d[2]^2+d[3]^2, 0, 0, -2*d[3]*d[4], 0,
>  -2*d[2]*d[4], -2*d[1]*d[4]],
>  [0, 0, d[1]*d[2], -d[1]*d[2], d[3]^2-d[4]^2, -d[1]*d[3], 0, -d[2]*d[3],
>  d[1]*d[4], d[2]*d[4]],
>  [d[2]*d[3], 0, 0, -d[2]*d[3],-d[1]*d[3], d[1]^2-d[4]^2, d[2]*d[4],
>  -d[1]*d[2], d[3]*d[4], 0],
>  [d[3]*d[4], d[3]*d[4], 0, 0, 0, -d[2]*d[4], d[1]^2+d[2]^2, -d[1]*d[4],
>  -d[2]*d[3], -d[1]*d[3]],
>  [0, d[1]*d[3], 0, -d[1]*d[3], -d[2]*d[3], -d[1]*d[2], d[1]*d[4],
>  d[2]^2-d[4]^2, 0, d[3]*d[4]],
>  [d[2]*d[4], 0, d[2]*d[4], 0, -d[1]*d[4], -d[3]*d[4], -d[2]*d[3], 0,
>  d[1]^2+d[3]^2, -d[1]*d[2]],
>  [0, d[1]*d[4], d[1]*d[4], 0, -d[2]*d[4], 0, -d[1]*d[3], -d[3]*d[4],
>  -d[1]*d[2], d[2]^2+d[3]^2]]):
```

Let $M = A^{1\times 10}/(A^{1\times 10}\, R)$ be the A-module finitely presented by R. Let us first compute the codimension of M.

```
> Codimension(R,A);
```

$$0$$

We get $\mathrm{codim}_A(M) = 0$, i.e., $\dim_A(M) = 4$. Let us check that $j_A(M) = \mathrm{codim}_A(M)$.

```
> GradeNumber(R,A);
```

$$0$$

Let us now compute the grade filtration $\{M_i\}_{i=0,\ldots,4}$ of M.

```
> G:=GradeFiltrationByGenerators(R,A):
```

We get a 2-step filtration of M since G contains only 2 elements:

```
> nops(G);
```

$$2$$

The A-module $M_1 = t(M)$ is defined by the residue classes of the rows of the first matrix of the first entry G_1 of G defined by:

```
> G[1];
```

$$\left[\begin{bmatrix}
0 & 0 & -d_2d_4 & 0 & 0 & d_3d_4 & d_2d_3 & 0 & -{d_3}^2 & 0\\
0 & -d_3d_4 & 0 & 0 & 0 & d_2d_4 & -{d_2}^2 & 0 & d_2d_3 & 0\\
0 & 0 & 0 & d_2d_3 & 0 & {d_4}^2 & -d_2d_4 & 0 & -d_3d_4 & 0\\
0 & -{d_4}^2 & -{d_4}^2 & -{d_2}^2-{d_3}^2 & 0 & 0 & 2\,d_3d_4 & 0 & 2\,d_2d_4 & 0\\
0 & -{d_4}^2 & 0 & -{d_2}^2 & 0 & 0 & 0 & 0 & 2\,d_2d_4 & 0\\
0 & 0 & -{d_4}^2 & -{d_3}^2 & 0 & 0 & 2\,d_3d_4 & 0 & 0 & 0\\
0 & d_3d_4 & 0 & 0 & 0 & -d_2d_4 & {d_2}^2 & 0 & -d_2d_3 & 0\\
0 & {d_4}^2 & 0 & {d_2}^2 & 0 & 0 & 0 & 0 & -2\,d_2d_4 & 0\\
0 & {d_4}^2 & {d_4}^2 & {d_2}^2+{d_3}^2 & 0 & 0 & -2\,d_3d_4 & 0 & -2\,d_2d_4 & 0\\
0 & 0 & d_1d_4 & 0 & 0 & 0 & -d_1d_3 & -d_3d_4 & 0 & {d_3}^2\\
0 & 0 & 0 & -d_1d_3 & 0 & 0 & d_1d_4 & -{d_4}^2 & 0 & d_3d_4\\
0 & 0 & {d_4}^2 & {d_3}^2 & 0 & 0 & -2\,d_3d_4 & 0 & 0 & 0\\
0 & 0 & -d_2d_4 & 0 & 0 & d_3d_4 & d_2d_3 & 0 & -{d_3}^2 & 0\\
0 & 0 & -d_1d_4 & 0 & 0 & 0 & d_1d_3 & d_3d_4 & 0 & -{d_3}^2\\
0 & 0 & 0 & d_1d_2 & {d_4}^2 & 0 & 0 & 0 & -d_1d_4 & -d_2d_4\\
0 & 0 & 0 & d_2d_3 & 0 & {d_4}^2 & -d_2d_4 & 0 & -d_3d_4 & 0\\
0 & 0 & 0 & d_1d_3 & 0 & 0 & -d_1d_4 & {d_4}^2 & 0 & -d_3d_4\\
0 & 0 & 0 & d_1d_2 & {d_4}^2 & 0 & 0 & 0 & -d_1d_4 & -d_2d_4\\
0 & 0 & 0 & 0 & d_3d_4 & -d_1d_4 & d_1d_2 & 0 & 0 & -d_2d_3\\
0 & 0 & 0 & 0 & 0 & -d_1d_4 & 0 & d_2d_4 & d_1d_3 & -d_2d_3
\end{bmatrix}, 1\right]$$

In other words, we have $t(M) = M_1 = (A^{1\times 20}\, G_{11})/(A^{1\times 10}\, R)$, where G_{11} is the first entry of G_1. Since we have

```
> G[2];
```

$$[[], 2]$$

we get $M_2 = 0$, which shows that the grade filtration of M is $0 = M_2 \subseteq M_1 \subseteq M$.

Let us now compute a presentation of the pure A-modules M/M_1 and M_1.

```
> F:=PureFactors(R,A):
```

Let us check whether or not the A-module M_i/M_{i+1} is 0 or i-pure for $i = 0, 1, 2$.

```
> map(IsPure,F,A);
```

$$[0, 1, \infty]$$

We obtain that 0-pure A-module $M/M_1 = M/t(M)$ is finitely presented by the following matrix:

```
> F[1];
```

$$\begin{bmatrix}
{d_2}^2 & {d_1}^2 & 0 & 0 & -2\,d_1d_2 & 0 & 0 & 0 & 0 & 0 \\
{d_3}^2 & 0 & {d_1}^2 & 0 & 0 & 0 & 0 & -2\,d_1d_3 & 0 & 0 \\
{d_4}^2 & 0 & 0 & {d_1}^2 & 0 & 0 & 0 & 0 & 0 & -2\,d_1d_4 \\
0 & {d_3}^2 & {d_2}^2 & 0 & 0 & -2\,d_2d_3 & 0 & 0 & 0 & 0 \\
0 & {d_4}^2 & 0 & {d_2}^2 & 0 & 0 & 0 & 0 & -2\,d_2d_4 & 0 \\
0 & 0 & {d_4}^2 & {d_3}^2 & 0 & 0 & -2\,d_3d_4 & 0 & 0 & 0 \\
d_2d_3 & 0 & 0 & 0 & -d_1d_3 & {d_1}^2 & 0 & -d_1d_2 & 0 & 0 \\
d_2d_4 & 0 & 0 & 0 & -d_1d_4 & 0 & 0 & 0 & {d_1}^2 & -d_1d_2 \\
d_3d_4 & 0 & 0 & 0 & 0 & 0 & {d_1}^2 & -d_1d_4 & 0 & -d_1d_3 \\
0 & d_1d_3 & 0 & 0 & -d_2d_3 & -d_1d_2 & 0 & {d_2}^2 & 0 & 0 \\
0 & d_1d_4 & 0 & 0 & -d_2d_4 & 0 & 0 & 0 & -d_1d_2 & {d_2}^2 \\
0 & d_3d_4 & 0 & 0 & 0 & -d_2d_4 & {d_2}^2 & 0 & -d_2d_3 & 0 \\
0 & 0 & d_1d_2 & 0 & {d_3}^2 & -d_1d_3 & 0 & -d_2d_3 & 0 & 0 \\
0 & 0 & d_1d_4 & 0 & 0 & 0 & -d_1d_3 & -d_3d_4 & 0 & {d_3}^2 \\
0 & 0 & d_2d_4 & 0 & 0 & -d_3d_4 & -d_2d_3 & 0 & {d_3}^2 & 0 \\
0 & 0 & 0 & d_1d_2 & {d_4}^2 & 0 & 0 & 0 & -d_1d_4 & -d_2d_4 \\
0 & 0 & 0 & d_1d_3 & 0 & 0 & -d_1d_4 & {d_4}^2 & 0 & -d_3d_4 \\
0 & 0 & 0 & d_2d_3 & 0 & {d_4}^2 & -d_2d_4 & 0 & -d_3d_4 & 0 \\
0 & 0 & 0 & 0 & 0 & d_1d_4 & 0 & -d_2d_4 & -d_1d_3 & d_2d_3 \\
0 & 0 & 0 & 0 & d_3d_4 & 0 & d_1d_2 & -d_2d_4 & -d_1d_3 & 0
\end{bmatrix}$$

Moreover, the 1-pure A-module $M_1 = t(M)$ is finitely presented by the matrix F_2. Since F_2 is a large matrix, let us print it in pieces.

```
> with(linalg):
> p:=coldim(F[2]);
```

$$p := 10$$

```
> q:=rowdim(F[2]);
```

$$q := 25$$

```
> submatrix(F[2],1..q,1..5);
```

$$\begin{bmatrix} 0 & -d_1 & 0 & d_4 & 0 \\ 0 & -d_2 & d_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ d_4 & 0 & d_2 & 0 & -d_3 \\ 0 & 0 & 0 & d_2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -d_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & d_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & d_2 \\ 0 & 0 & d_3 & 0 & 0 \\ 0 & 0 & 0 & d_3 & 0 \\ 0 & 0 & 0 & 0 & d_1 \\ 0 & -d_4 & d_2 & d_1 & 0 \\ d_3 & 0 & 0 & 0 & -d_4 \\ -d_2 & 0 & -d_4 & 0 & 0 \\ 0 & -d_3 & 0 & 0 & 0 \\ {d_4}^2 - {d_3}^2 & 0 & d_2 d_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & {d_2}^2 - {d_4}^2 & 0 & 0 \\ 0 & 0 & 0 & {d_1}^2 - {d_4}^2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

```
> submatrix(F[2],1..q,6..p);
```

$$\begin{bmatrix} 0 & -d_3 & 0 & 0 & 0 \\ -d_3 & 0 & 0 & 0 & 0 \\ d_1 & 0 & -d_3 & d_4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & d_4 & -d_3 & 0 \\ 0 & d_2 & -d_3 & d_4 & -d_4 \\ 0 & d_3 & d_2 & 0 & 0 \\ 0 & 0 & d_4 & -d_3 & d_3 \\ d_4 & 0 & 0 & d_1 & 0 \\ -d_4 & 0 & 0 & 0 & -d_1 \\ 0 & -d_4 & 0 & 0 & d_2 \\ 0 & -d_4 & 0 & -d_2 & d_2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & d_1 & 0 & 0 & 0 \\ 0 & 0 & -d_1 & 0 & 0 \\ d_2 & d_1 & 0 & 0 & 0 \\ 0 & -d_1 d_3 & 0 & 0 & 0 \\ 0 & 0 & 2\, d_3 d_4 & -{d_4}^2 - {d_3}^2 & -{d_2}^2 + {d_4}^2 \\ 0 & 0 & 0 & {d_1}^2 + {d_2}^2 + {d_3}^2 - {d_4}^2 & 0 \\ d_3 d_4 & 0 & -d_1 d_4 & d_1 d_3 & 0 \\ 0 & d_3 d_4 & -d_2 d_4 & d_2 d_3 & -d_2 d_3 \\ 0 & 0 & 0 & 0 & {d_1}^2 + {d_2}^2 + {d_3}^2 - {d_4}^2 \\ 0 & 0 & {d_1}^2 + {d_2}^2 + {d_3}^2 - {d_4}^2 & 0 & 0 \\ {d_2}^2 + {d_3}^2 - {d_4}^2 & 0 & d_1 d_3 & -d_1 d_4 & 0 \\ 0 & {d_1}^2 + {d_3}^2 - {d_4}^2 & d_2 d_3 & -d_2 d_4 & d_2 d_4 \end{bmatrix}$$

6.2 Equidimensional decomposition of affine algebraic varieties

Example 13. Let us consider the commutative polynomial ring $A = \mathbb{Q}[x, y, z]$

```
> A:=DefineOreAlgebra(diff=[x,s1],diff=[y,s2],diff=[z,s3],polynom=[s1,s2,s3]):
```

and the matrix $R \in A^{1\times 3}$ defined by:

```
> R:=evalm([[x^3+x^2*y+x^2*z-x^2-x*z-y*z-z^2+z],[x^2*y*z+x^2*y-y*z^2-y*z],
> [x^2*y^2-x^2*y-y^2*z+y*z]]);
```

$$R := \begin{bmatrix} x^3 + x^2y + x^2z - x^2 - xz - yz - z^2 + z \\ x^2yz + x^2y - yz^2 - yz \\ x^2y^2 - x^2y - y^2z + yz \end{bmatrix}$$

Let us consider the A-module $M = A/I$, where $I = A^{1\times 3}\, R$ is the ideal of A generated by the three entries of R. The A-module M was first considered in Exercise 4.4.5 of G.-M. Greuel, G. Pfister, “A **Singular** Introduction to Commutative Algebra”, Springer, 2002, p. 261.

Let us first try to solve the polynomial system defined by I using the Maple command `solve`:

```
> solve(convert(R,set));
```

$$\left\{x = x, y = y, z = x^2\right\}, \left\{x = RootOf\left(\_Z^2 + 1\right), y = y, z = -1\right\},$$
$$\{x = x, y = 0, z = -x + 1\}, \{x = 1, y = 1, z = -1\}$$

The Maple output shows that the complex algebraic variety $V(I)$ defined by the ideal I is formed by a point, 3 curves and a hypersurface. In particular, $V(I)$ is not equidimensional. Hence, let us check again that M is not a pure A-module.

```
> IsPure(R,A);
```

$$\mathit{false}$$

Let us now compute the grade filtration of M.

```
> G:=GradeFiltrationByGenerators(R,A);
```

$$G := [[\left[\begin{array}{c} 1 \end{array}\right], 1], [\left[\begin{array}{c} x^2 - z \end{array}\right], 2], [\left[\begin{array}{c} x^2 y - yz \end{array}\right], 3]]$$

If $\pi \colon A \longrightarrow M$ is the canonical projection onto M and $u = \pi(1)$ the generator of M, then we have $M_1 = A\, u = M$, $M_2 = A\,(x^2 - z)\, u$, $M_3 = A\,(y\,(x^2 - z)\, u)$, and $M_4 = 0$.

If an option is added to the command `GradeFiltrationByGenerators`, then the annihilator of the generators of the A-modules M_i's are also computed and returned in the first matrix of each entry of the output.

```
> H:=GradeFiltrationByGenerators(R,A,opt);
```

$$H := [[\left[\begin{array}{c} x^3 + x^2 y + x^2 z - x^2 - xz - yz - z^2 + z \\ x^2 yz + x^2 y - yz^2 - yz \\ x^2 y^2 - x^2 y - y^2 z + yz \end{array}\right], \left[\begin{array}{c} 1 \end{array}\right], 1], [\left[\begin{array}{c} x + y + z - 1 \\ yz + y \\ y^2 - y \end{array}\right], \left[\begin{array}{c} x^2 - z \end{array}\right], 2],$$
$$[\left[\begin{array}{c} z + 1 \\ y - 1 \\ x - 1 \end{array}\right], \left[\begin{array}{c} x^2 y - yz \end{array}\right], 3]]$$

Hence, we get $M_1 \cong A/(A^{1\times 3}\, H_{11}) = M$, $M_2 \cong A/(A^{1\times 3}\, H_{21})$, and $M_3 \cong A/(A^{1\times 3}\, H_{31})$, where H_{i1} is the first matrix in the i^{th} entry of H.

Another way to define the grade filtration $\{M_i\}_{i=0,\ldots,3}$ of M is by means of finitely presented A-modules $L_i \cong M_i$ and injective $\theta_i \in \hom_A(L_i, M)$ for $i = 1, 2, 3$ (see Algorithm 3).

```
> J:=GradeFiltrationByMorphisms(R,A);
```

$$J := [[\left[\begin{array}{c} x^2 y^2 - x^2 y - y^2 z + yz \\ x^2 yz + x^2 y - yz^2 - yz \\ x^3 + x^2 y + x^2 z - x^2 - xz - yz - z^2 + z \end{array}\right], \left[\begin{array}{c} 1 \end{array}\right]], [\left[\begin{array}{c} y^2 - y \\ yz + y \\ x + y + z - 1 \end{array}\right], \left[\begin{array}{c} x^2 - z \end{array}\right]],$$
$$[\left[\begin{array}{c} y - 1 \\ z + 1 \\ x - 1 \end{array}\right], \left[\begin{array}{c} y\left(x^2 - z\right) \end{array}\right]]]$$

We obtain $L_1 = A/(A^{1\times 3}\, J_{11})$, $L_2 = A/(A^{1\times 3}\, J_{21})$, and $L_3 = A/(A^{1\times 3}\, J_{31})$, where J_{i1} is the first matrix of the i^{th} entry of J. Moreover, the injective A-homomorphism $\theta_i \colon L_i \longrightarrow M$ is defined by $\theta_i(\rho_i'(\lambda)) = \pi(\lambda\, J_{i2})$, where J_{i2} is the second matrix in the i^{th} entry of J and ρ_i' is the canonical projection onto L_i. We find $L_1 \cong M_1$, $L_2 \cong M_2$, and $L_3 \cong M_3$. Let us check again that the A-homomorphisms θ_i's are injective:

```
>  seq(TestInj(J[i][1],R,J[i][2],A),i=1..3)
```

$$true, true, true$$

Let us compute a finite presentation of the A-module M_i/M_{i+1} for $i = 1, 2, 3$.

```
>  F:=PureFactors(R,A);
```

$$F := \left[\left[\begin{array}{c} 1 \end{array}\right], \left[\begin{array}{c} x^2 - z \end{array}\right], \left[\begin{array}{c} y \\ x - 1 + z \end{array}\right], \left[\begin{array}{c} y - 1 \\ z + 1 \\ x - 1 \end{array}\right]\right]$$

We obtain $M/M_1 = A/(A\, F_1) = 0$, i.e., $M = M_1 = t(M)$, $M_1/M_2 = A/(A\, F_2)$, $M_2/M_3 = A/(A^{1\times 2}\, F_3)$, and $M_3 = A/(A^{1\times 3}\, F_3)$, where F_i is the i^{th} matrix of F.

Let us check again that the A-modules M_i/M_{i+1}'s are either 0 or i-pure.

```
>  map(IsPure,F,A);
```

$$[\infty, 1, 2, 3]$$

We find again that $M/M_1 = 0$ and M_1/M_2 (resp., M_2/M_3 and M_3) is 1 pure (resp., 2 and 3 pure).

Let us now compute a presentation of the A-module M based on the grade filtration of M.

```
>  P:=PurePresentation(R,A);
```

$$P := \left[\left[\begin{array}{c} x^3 + x^2 y + x^2 z - x^2 - x z - y z - z^2 + z \\ x^2 y z + x^2 y - y z^2 - y z \\ x^2 y^2 - x^2 y - y^2 z + y z \end{array}\right], \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & x^2 - z & -1 & 0 \\ 0 & 0 & y & -1 \\ 0 & 0 & x - 1 + z & 1 \\ 0 & 0 & 0 & y - 1 \\ 0 & 0 & 0 & z + 1 \\ 0 & 0 & 0 & x - 1 \end{array}\right],\right.$$

$$\left.\left[\begin{array}{cccc} 1 & 1 & 0 & 0 \end{array}\right], \left[\begin{array}{c} 0 \\ 1 \\ x^2 - z \\ x^2 y - y z \end{array}\right]\right]$$

We obtain $M = A/(A^{1\times 3}\, P_1) \cong \overline{M} = A^{1\times 4}/(A^{1\times 7}\, P_2)$, where P_i is the i^{th} matrix of P. If $\overline{\pi}$ is the canonical projection onto $\overline{M}$, then $\varphi \colon M \longrightarrow \overline{M}$ defined by $\varphi(\pi(\lambda)) = \overline{\pi}(\lambda\, P_3)$ is an isomorphism, whose inverse $\varphi^{-1} \colon \overline{M} \longrightarrow M$ is $\varphi^{-1}(\overline{\pi}(\mu)) = \pi(\mu\, P_4)$.

Let us check that φ is an isomorphism and φ^{-1} is defined by P_4.

```
> TestIso(P[1],P[2],P[3],A);
```

$$true$$

```
> TestIso(P[2],P[1],P[4],A);
```

$$true$$

Since $M_1 = t(M) = M$, we can simply compute a new presentation of M based on the grade filtration of $t(M)$.

```
> Q:=PurePresentationOfTorsionSubmodule_R(R,A);
```

$$Q := [\begin{bmatrix} x^2 - z & -1 & 0 \\ 0 & y & -1 \\ 0 & x - 1 + z & 1 \\ 0 & 0 & y - 1 \\ 0 & 0 & z + 1 \\ 0 & 0 & x - 1 \end{bmatrix}, \begin{bmatrix} x^3 + x^2 y + x^2 z - x^2 - xz - yz - z^2 + z \\ x^2 yz + x^2 y - yz^2 - yz \\ x^2 y^2 - x^2 y - y^2 z + yz \end{bmatrix}, \begin{bmatrix} 1 \\ x^2 - z \\ x^2 y - yz \end{bmatrix}]$$

We get $M_1 = M = A/(A^{1\times 3}\, Q_2) \cong \overline{M}_1 = A^{1\times 3}/(A^{1\times 6}\, Q_1)$, where this A-isomorphism is defined by the matrix of Q_3.

Finally, let us check again that Q_3 defines an isomorphism from $\overline{M}_1$ to M.

```
> TestIso(Q[1],Q[2],Q[3],A);
```

$$true$$

Example 14. Let us consider the commutative polynomial ring $A = \mathbb{Q}[x_1, x_2, x_3, x_4]$

```
> A:=DefineOreAlgebra(diff=[x[1],s1],diff=[x[2],s2],diff=[x[3],s3],diff=
> [x[4],s4],polynom=[s1,s2,s3,s4]):
```

and the matrix $R \in A^{1\times 3}$ defined by:

```
> R:=evalm([[x[1]^3],[x[2]^3],[(x[1]^2+x[2]^2)*x[4]+x[1]*x[2]*x[3]]]);
```

$$R := \begin{bmatrix} {x_1}^3 \\ {x_2}^3 \\ \left({x_1}^2 + {x_2}^2\right) x_4 + x_1 x_2 x_3 \end{bmatrix}$$

Let us consider the A-module $M = A/I$, where $I = A^{1\times 3}\, R$ is the ideal of A generated by the three entries of R, first considered in F. S. Macaulay, "The Algebraic Theory of Modular Systems", Cambridge 1994 (first published in 1916), p. 44.

Let us first try to solve the polynomial system defined by I using the Maple command `solve`:

```
> solve(convert(R,set));
```

$$\{x_1 = 0, x_2 = 0, x_3 = x_3, x_4 = x_4\}$$

According to Maple, the affine algebraic variety $V(I)$ defined by I is the 2-dimensional algebraic variety ($x_1 = 0$, $x_2 = 0$, $x_3 = x_3$, $x_4 = x_4$). In particular, if this result is correct, then $V(I)$ would be an equidimensional affine algebraic variety.

Let us compute the grade filtration of M.

```
> G:=GradeFiltrationByGenerators(R,A);
```

$$G := [[[\ 1\ ], 1], [[\ 1\ ], 2], [\begin{bmatrix} x_2 {x_1}^2 \\ -x_1 {x_2}^2 \\ x_1 {x_2}^2 \\ -x_2 {x_1}^2 \end{bmatrix}, 3], [[\ -{x_1}^2 {x_2}^2\ ], 4]]$$

If $\pi\colon A \longrightarrow M$ is the canonical projection onto M and $u = \pi(1)$ the generator of M, then $M_1 = A\,u = M$, $M_2 = A\,u = M$, $M_3 = A\,(x_2\,x_1^2)\,u + A\,(x_1\,x_2^2)\,u$, $M_4 = A\,(x_1^2\,x_2^2)\,u$, and $M_5 = 0$. Hence, the command `solve` does not compute the whole solution set of the polynomial system defined by I. In particular, $V(I)$ is not an equidimensional affine algebraic variety.

Let us now compute a finite presentation of the A-module M_i/M_{i+1} for $i = 0, \ldots, 4$.

```
> F:=PureFactors(R,A);
```

$$F := [[\ 1\ ], [\ 1\ ], \begin{bmatrix} {x_1}^3 \\ {x_2}^3 \\ x_1 {x_2}^2 \\ x_2 {x_1}^2 \\ x_4 {x_1}^2 + x_4 {x_2}^2 + x_1 x_2 x_3 \end{bmatrix}, \begin{bmatrix} 0 & -x_2 \\ x_2 & 0 \\ -x_4 & -x_3 \\ 0 & -x_1 \\ x_1 & 0 \\ -x_3 & -x_4 \\ {x_3}^2 - {x_4}^2 & 0 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}]$$

We get $M/M_1 = A/(A\,F_1) = 0$, $M_1/M_2 = A/(A\,F_2) = 0$, $M_2/M_3 = A/(A^{1\times 5}\,F_3)$, $M_3/M_4 = A^{1\times 2}/(A^{1\times 7}\,F_4)$, and $M_4 = A/(A^{1\times 4}\,F_5)$.

Let us now check that the A-modules M_i/M_{i+1}'s are either 0 or i-pure.

```
> map(IsPure,F,A);
```

$$[\infty, \infty, 2, 3, 4]$$

Let us compute a new presentation of the A-module M based on the grade filtration of M.

```
> P:=PurePresentation(R,A):
```

We obtain that the A-module M finitely presented by the matrix

```
> P[1];
```

$$\begin{bmatrix} {x_1}^3 \\ {x_2}^3 \\ \left({x_1}^2 + {x_2}^2\right) x_4 + x_1 x_2 x_3 \end{bmatrix}$$

is isomorphic to the A-module $\overline{M}$ finitely presented by the matrix

```
> P[2];
```

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & {x_1}^3 & 0 & 0 & 0 \\ 0 & 0 & {x_2}^3 & 0 & 0 & 0 \\ 0 & 0 & x_1{x_2}^2 & 0 & -1 & 0 \\ 0 & 0 & {x_1}^2x_2 & -1 & 0 & 0 \\ 0 & 0 & x_4{x_1}^2+x_4{x_2}^2+x_1x_2x_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -x_2 & 0 \\ 0 & 0 & 0 & x_2 & 0 & -1 \\ 0 & 0 & 0 & -x_4 & -x_3 & 0 \\ 0 & 0 & 0 & 0 & -x_1 & 1 \\ 0 & 0 & 0 & x_1 & 0 & 0 \\ 0 & 0 & 0 & -x_3 & -x_4 & 0 \\ 0 & 0 & 0 & {x_3}^2-{x_4}^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_1 \\ 0 & 0 & 0 & 0 & 0 & x_2 \\ 0 & 0 & 0 & 0 & 0 & x_3 \\ 0 & 0 & 0 & 0 & 0 & x_4 \end{bmatrix}$$

i.e., $M \cong \overline{M} = A^{1\times 6}/(A^{1\times 18}\,P_2)$. Moreover, $\varphi \colon M \longrightarrow \overline{M}$ defined by $\varphi(\pi(\lambda)) = \overline{\pi}(\lambda\,P_3)$, where the matrix P_3 is given by

```
>  P[3];
```

$$\begin{bmatrix} 1 & 1 & -1 & 0 & 0 & 0 \end{bmatrix}$$

and $\overline{\pi} \colon A^{1\times 6} \longrightarrow \overline{M}$ is the canonical projection onto $\overline{M}$, is an A-isomorphism. Its inverse $\varphi^{-1} \colon \overline{M} \longrightarrow A^{1\times 6}$ is defined by $\varphi^{-1}(\overline{\pi}(\mu)) = \pi(\mu\,P_4)$, where the matrix P_4 is defined by:

```
>  P[4];
```

$$\begin{bmatrix} 0 \\ 0 \\ -1 \\ -{x_1}^2x_2-{x_2}^3 \\ -x_1{x_2}^2 \\ -{x_1}^2{x_2}^2-{x_2}^4 \end{bmatrix}$$

Finally, let us check again that φ is an isomorphism

```
>  TestIso(P[1],P[2],P[3],A);
```

true

and φ^{-1} is defined by matrix P_4.

```
> TestIso(P[2],P[1],P[4],A);
```

$$true$$

6.3 Integration of linear PD systems

Example 15. Let A be the ring of PD operators in $dx = \frac{\partial}{\partial x}$ and $dt = \frac{\partial}{\partial t}$ with coefficients in $\mathbb{Q}[x, t]$

```
> A:=DefineOreAlgebra(diff=[dx,x],diff=[dt,t],polynom=[x,t]):
```

and the matrix $R \in A^{1\times 2}$ of PD operators defined by:

```
> R:=evalm([[dt^2*(dx-dt)],[dt*dx*(dx-dt)]]);
```

$$\begin{bmatrix} dt^2\,(dx - dt) \\ dt\;dx\;(dx - dt) \end{bmatrix}$$

The corresponding linear system $R\,y(t, x) = 0$ is defined by the following equations:

```
> Eqs:=map(a->a=0,convert(ApplyMatrix(R,[y(t,x)],A),set));
```

$$Eqs := \left\{-\frac{\partial^3}{\partial x \partial t^2} y\,(t, x) + \frac{\partial^3}{\partial x^2 \partial t} y\,(t, x) = 0,\, \frac{\partial^3}{\partial x \partial t^2} y\,(t, x) - \frac{\partial^3}{\partial t^3} y\,(t, x) = 0\right\}$$

Let us use the Maple command `pdsolve` to integrate the above linear PD system.

```
> st:=time():  sol:=pdsolve(Eqs,y(t,x)); time()-st;
```

Error, (in combine/power) too many levels of recursion
28.679

Maple cannot solve the linear PD system due to bugs!

Let us now study the grade filtration of the A-module $M = A/(A^{1\times 2}\,R)$.

```
> G:=GradeFiltrationByGenerators(R,A,opt);
```

$$G := [[\begin{bmatrix} dt^2\,dx - dt^3 \\ -dt^3 + dt\;dx^2 \end{bmatrix}, \begin{bmatrix} 1 \end{bmatrix}, 1], [\begin{bmatrix} dt \\ dx \end{bmatrix}, \begin{bmatrix} -dt\;dx + dt^2 \end{bmatrix}, 2]]$$

If $\pi\colon A \longrightarrow M$ be the canonical projection onto M and $u = \pi(1)$ the generator of A, then $M_1 = A\,u = A/(A^{1\times 2}\,R) = M$, $M_2 = (-dt\,dx + dt^2)\,u \cong A/(A\,dt + A\,dx)$, and $M_3 = 0$.

Let us now compute a finite presentation of the A-module M_i/M_{i+1} for $i = 0, 1, 2$.

```
> F:=PureFactors(R,A);
```

$$F := [\begin{bmatrix} 1 \end{bmatrix}, \begin{bmatrix} dt\;dx - dt^2 \end{bmatrix}, \begin{bmatrix} dt \\ dx \end{bmatrix}]$$

We obtain $M/M_1 = A/(A\,F_1)$, $M_1/M_2 = A/(A\,F_2) = 0$, and $M_2 = A/(A^{1\times 2}\,F_3)$, where F_i is the i^{th} matrix of F.

Let us check whether or not the A-module M_i/M_{i+1} is either 0 or i-pure for $i = 0, 1, 2$.

```
> map(IsPure,F,A);
```

$$[\infty, 1, 2]$$

Let us now compute a finite presentation of M based on the grade filtration of M.

```
> P:=PurePresentation(R,A);
```

$$P := \left[\begin{bmatrix} dt^2\,(dx - dt) \\ dt\,dx\,(dx - dt) \end{bmatrix}, \begin{bmatrix} 1 & -1 & 0 \\ 0 & dt\,dx - dt^2 & 1 \\ 0 & 0 & dt \\ 0 & 0 & dx \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -dt\,dx + dt^2 \end{bmatrix} \right]$$

We obtain $M = A/(A^{1\times 2}\,P_1) \cong \overline{M} = A^{1\times 3}/(A^{1\times 4}\,P_2)$, where P_i is the i^{th} matrix of P. Moreover, the A-isomorphism $\varphi\colon M \longrightarrow \overline{M}$ is defined by $\varphi(\pi(\lambda)) = \overline{\pi}(\lambda\,P_3)$, where $\overline{\pi}\colon A^{1\times 3} \longrightarrow \overline{M}$ is the canonical projection onto $\overline{M}$. Finally, $\varphi^{-1}\colon \overline{M} \longrightarrow M$ is defined by $\varphi^{-1}(\overline{\pi}(\mu)) = \pi(\mu\,P_4)$.

Let us now check again that φ is an isomorphism and φ^{-1} is defined by P_4.

```
> TestIso(P[1],P[2],P[3],A);
```

$$true$$

```
> TestIso(P[2],P[1],P[4],A);
```

$$true$$

Let us now try to integrate the above linear PD system by using its equivalence form $P_2\,z = 0$.

```
> iv:=op(A[3]);
```

$$iv := x, t$$

```
> Eqs:=map(a->a=0,convert(convert(ApplyMatrix(P[2],[zeta[1](iv),zeta[2](iv),
> zeta[3](iv)],A),vector),set));
```

$$Eqs :=$$

$$\left\{\zeta_1(x,t) - \zeta_2(x,t) = 0, -\frac{\partial^2}{\partial t^2}\zeta_2(x,t) + \frac{\partial^2}{\partial x \partial t}\zeta_2(x,t) + \zeta_3(x,t) = 0, \frac{\partial}{\partial t}\zeta_3(x,t) = 0, \frac{\partial}{\partial x}\zeta_3(x,t) = 0\right\}$$

We obtain

```
> st:=time():  z:=pdsolve(Eqs,zeta[1](iv),zeta[2](iv),zeta[3](iv));
> time()-st;
```

$$z := \{\zeta_1(x,t) = \_F1(x) + \_F2(x+t) - 1/2\,\_C1\,x\,(x+2\,t),$$
$$\zeta_2(x,t) = \_F1(x) + \_F2(x+t) - 1/2\,\_C1\,x\,(x+2\,t), \zeta_3(x,t) = \_C1\}$$

$$0.019$$

i.e., the general solution Z of $P_2\,z = 0$ is defined by:

```
> Z:=evalm([[rhs(sol[1])],[rhs(sol[2])],[rhs(sol[3])]]);
```

$$Z := \begin{bmatrix} \_F1(x) + \_F2(x+t) - 1/2\,\_C1\,x\,(x+2\,t) \\ \_F1(x) + \_F2(x+t) - 1/2\,\_C1\,x\,(x+2\,t) \\ \_C1 \end{bmatrix}$$

Let us check again that Z is a solution of the linear PD system $P_2\,Z = 0$:

```
> ApplyMatrix(P[2],Z,A);
```

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now, the solution of the linear PD system $R\,y = 0$ is defined by $y = P_3\,z$, i.e.:

```
> y:=ApplyMatrix(P[3],Z,A);
```

$$y := \left[\ \_F1\,(x) + \_F2\,(x+t) - 1/2\,\_C1\,x^2 - \_C1\,xt\ \right]$$

Let us check that y is a solution of the linear PD system $R\,y = 0$.

```
> ApplyMatrix(R,y,A);
```

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This result can be directly obtained by the command `IntegrationOfDModule`.

```
> sol:=IntegrationOfDModule(R,A);
```

$$sol := \left[\ \_F1\,(x) + \_F2\,(x+t) - 1/2\,\_C1\,x^2 - \_C1\,xt\ \right]$$

```
> ApplyMatrix(R,sol,A);
```

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Example 16. Let A be the ring of PD operators in $dx = \frac{\partial}{\partial x}$ and $dt = \frac{\partial}{\partial t}$ with coefficients in $\mathbb{Q}[x,t]$

```
> A:=DefineOreAlgebra(diff=[dx,x],diff=[dt,t],polynom=[x,t]):
```

and the matrix $R \in A^{1\times 2}$ of PD operators defined by:

```
> R:=evalm([[dx^2*(dt-dx)],[dt^2*(dt-dx)]]);
```

$$R := \begin{bmatrix} dx^2\,(dt-dx) \\ dt^2\,(dt-dx) \end{bmatrix}$$

Let us use the Maple command `pdsolve` to integrate the linear PD system $R\,y(x,t) = 0$, i.e.,

```
> iv:=op(A[3]);
```

$$iv := x, t$$

the linear system of PD equations defined by:

```
> Eqs:=map(a->a=0,convert(ApplyMatrix(R,[y(iv)],A),set));
```

$$Eqs := \left\{-\frac{\partial^3}{\partial x \partial t^2} y\,(x,t) + \frac{\partial^3}{\partial t^3} y\,(x,t) = 0, \frac{\partial^3}{\partial x^2 \partial t} y\,(x,t) - \frac{\partial^3}{\partial x^3} y\,(x,t) = 0\right\}$$

Maple cannot integrate the linear PD system due to bugs!

```
> st:=time():  sol:=pdsolve(Eqs,y(iv)); time()-st;
```

Error, (in dchange/funcs) not implemented case of many integrals w.r.t the same variable inside a multiple integral

0.698

Let us now compute the grade filtration of the A-module $M = A/(A^{1\times 2}\, R)$.

```
> G:=GradeFiltrationByGenerators(R,A);
```

$$G := [[\begin{bmatrix} -dt^3 + dt^2\,dx \\ -dx^2\,dt + dx^3 \end{bmatrix}, \begin{bmatrix} 1 \end{bmatrix}, 1], [\begin{bmatrix} dt^2 \\ dx^2 \end{bmatrix}, \begin{bmatrix} dt - dx \end{bmatrix}, 2]]$$

If $\pi\colon A \longrightarrow M$ is the canonical projection onto M and $u = \pi(1)$ the generator of M, then $M_1 = A\,u = M = A/(A^{1\times 2}\, R)$, $M_2 = A\,(dt - dx)\,u \cong A/(A\,dt^2 + A\,dx^2)$, and $M_3 = 0$.

Let us compute a finite presentation of the A-module M_i/M_{i+1} for $i = 0, 1, 2$.

```
> F:=PureFactors(R,A);
```

$$F := \begin{bmatrix} 1 \end{bmatrix}, \begin{bmatrix} dx - dt \end{bmatrix}, \begin{bmatrix} dt^2 \\ dx^2 \end{bmatrix}]$$

We get $M/M_1 = A/(A\,F_1) = 0$, $M_1/M_2 = A/(A\,F_2)$, and $M_2 = A/(A^{1\times 2}\, F_3)$, where F_i is the i^{th} matrix of F.

Let us check again that M_i/M_{i+1} is either 0 or i-pure for $i = 0, 1, 2$.

```
> map(IsPure,F,A);
```

$$[\infty, 1, 2]$$

Since $M = M_1 = t(M)$, M is a torsion A-module. Let us compute a finite presentation of M based on the grade filtration of M_1.

```
> P:=PurePresentationOfTorsionSubmodule(R,A);
```

$$P := [\begin{bmatrix} dx - dt & 1 \\ 0 & dt^2 \\ 0 & dx^2 \end{bmatrix}, \begin{bmatrix} dx^2\,(dt - dx) \\ dt^2\,(dt - dx) \end{bmatrix}, \begin{bmatrix} 1 \\ dt - dx \end{bmatrix}]$$

We get $L = A^{1\times 2}/(A^{1\times 3}\, P_1) \subseteq M = A/(A^{1\times 2}\, P_2)$. The injection $\iota : L \longrightarrow M$ is defined by $\iota(\kappa(\mu)) = \pi(\mu\, P_3)$, where P_i is the i^{th} matrix of P and $\kappa\colon A^{1\times 2} \longrightarrow L$ is the canonical projection onto L. Let us check again that ι is an injection.

```
> TestInj(P[1],P[2],P[3],A);
```

true

Since $M_1 = M$, ι is also an isomorphism, which can be easily check again.

```
> TestIso(P[1],P[2],P[3],A);
```

true

The inverse $\iota^{-1} : M \longrightarrow L$ of ι can then be computed as follows.

```
> T:=InverseMorphism(P[1],P[2],P[3],A);
```

$$T := [\begin{bmatrix} 1 & 0 \end{bmatrix}, \begin{bmatrix} -dx^2 & 0 & 1 \\ -dt^2 & 1 & 0 \end{bmatrix}]$$

Thus, $\iota^{-1}\colon M \longrightarrow L$ is defined by $\iota^{-1}(\pi(\lambda)) = \kappa(\lambda\, T_1)$, where T_1 is the first matrix of T. Let us check again that the A-homomorphism defined by T_1 defined an isomorphism.

```
> TestIso(P[2],P[1],T[1],A);
```

$$true$$

Let us try to integrate $P_1\, z = 0$ using the command `IntegrationOfTorsionDSubmodule`.

```
> z:=IntegrationOfTorsionDSubmodule(R,A);
```

$$z := \left[\begin{array}{c} -1/6\, x^3\_C3 + (-1/2\,\_C1 - 1/2\,\_C3\, t - 1/2\,\_C4)\, x^2 + (-\_C1\, t - \_C2)\, x + \_F1\,(t+x) \\ (x\_C3 + \_C1)\, t + x\_C4 + \_C2 \end{array}\right]$$

Let us check again that z is a solution of $P_1\, z = 0$.

```
> ApplyMatrix(P[1],z,A);
```

$$\left[\begin{array}{c} 0 \\ 0 \\ 0 \end{array}\right]$$

Then, $y = T_1\, z$, namely,

```
> y:=ApplyMatrix(T[1],z,A);
```

$$y := \left[\begin{array}{c} -1/6\, t^3\_C3 - 1/2\, t^2\_C3\, x - 1/2\, t^2\_C4 - 1/2\,\_C1\, t^2 - \_C1\, tx - \_C2\, t + \_F1\,(x+t) \end{array}\right]$$

is a solution of the linear PD system $R\, y = 0$.

```
> ApplyMatrix(R,y,A);
```

$$\left[\begin{array}{c} 0 \\ 0 \end{array}\right]$$

This last result can be directly be obtained using the command `IntegrationOfDModule`.

```
> y:=IntegrationOfDModule(R,A);
```

$$y := \left[\begin{array}{c} -1/6\, x^3\_C3 - 1/2\, x^2\_C3\, t - 1/2\, x^2\_C4 - 1/2\,\_C1\, x^2 - \_C1\, xt - \_C2\, x + \_F1\,(t+x) \end{array}\right]$$

```
> ApplyMatrix(R,y,A);
```

$$\left[\begin{array}{c} 0 \\ 0 \end{array}\right]$$

Example 17. Let A be the ring of PD operators in $d_1 = \frac{\partial}{\partial x_1}$, $d_2 = \frac{\partial}{\partial x_2}$, and $d_3 = \frac{\partial}{\partial x_3}$ with coefficients in the ring $\mathbb{Q}[x_1, x_2, x_3]$

```
> A:=DefineOreAlgebra(diff=[d[1],x[1]],diff=[d[2],x[2]],polynom=[x[1],x[2]]):
```

and the matrix $R \in A^{1\times 2}$ of PD operators defined by:

```
> R:=evalm([[d[1]^2+d[2]*d[1]-(x[1]+x[2])*d[1]-1],[d[2]^2+d[2]*d[1]
> -(x[1]+x[2])*d[2]-1]]);
```

$$R := \left[\begin{array}{c} {d_1}^2 + d_2\, d_1 - (x_1 + x_2)\, d_1 - 1 \\ {d_2}^2 + d_2\, d_1 - (x_1 + x_2)\, d_2 - 1 \end{array}\right]$$

Let us try to integrate the linear PD system $R\,\eta = 0$ defined by

```
> iv:=op(A[3]);
```

$$iv := x_1, x_2$$

```
> Eqs:=map(a->a=0,convert(convert(ApplyMatrix(R,[eta(iv)],A),vector),set));
```

$$Eqs :=$$

$$\Big\{-\eta(x_1,x_2) - \left(\frac{\partial}{\partial x_1}\eta(x_1,x_2)\right)x_1 - \left(\frac{\partial}{\partial x_1}\eta(x_1,x_2)\right)x_2 + \frac{\partial^2}{\partial x_2 \partial x_1}\eta(x_1,x_2) + \frac{\partial^2}{\partial x_1^2}\eta(x_1,x_2) = 0,$$
$$-\eta(x_1,x_2) - \left(\frac{\partial}{\partial x_2}\eta(x_1,x_2)\right)x_1 - \left(\frac{\partial}{\partial x_2}\eta(x_1,x_2)\right)x_2 + \frac{\partial^2}{\partial x_2 \partial x_1}\eta(x_1,x_2) + \frac{\partial^2}{\partial x_2^2}\eta(x_1,x_2) = 0\Big\}$$

Maple cannot solve the linear PD system since no output is returned:

```
> eta:=pdsolve(Eqs,eta(iv));
```

$$eta :=$$

Let us now compute the grade filtration of the left A-module $M = A/(A^{1\times 2}\,R)$.

```
> GradeFiltrationByGenerators(R,A,opt);
```

$$[[\begin{bmatrix} {d_2}^2 + d_2 d_1 - d_2 x_1 - d_2 x_2 - 1 \\ d_2 x_2 - d_1 x_2 + d_2 x_1 - d_1 x_1 - {d_2}^2 + {d_1}^2 \end{bmatrix}, \begin{bmatrix} 1 \end{bmatrix}, 1], [\begin{bmatrix} d_2 \\ d_1 \end{bmatrix}, \begin{bmatrix} -d_1 - d_2 + x_1 + x_2 \end{bmatrix}, 2]]$$

If $\pi \colon A \longrightarrow M = A/(A^{1\times 2}\,R)$ is the canonical projection and $u = \pi(1)$ the generator of M, then $M_1 = A\,u = M = A/(A^{1\times 2}\,R)$, $M_2 = A\,(-d_1 - d_2 + x_1 + x_2)\,u \cong A/(A\,d_1 + A\,d_2)$, and $M_3 = 0$.

Let us compute a finite presentation of the left A-module M_i/M_{i+1} for $i = 0, 1, 2$.

```
> F:=PureFactors(R,A);
```

$$F := [\begin{bmatrix} 1 \end{bmatrix}, \begin{bmatrix} d_1 + d_2 - x_1 - x_2 \end{bmatrix}, \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}]$$

We obtain $M/M_1 = A/(A\,F_1) = 0$, $M_1/M_2 = A/(A\,F_2)$, and $M_2 = A/(A^{1\times 3}\,F_3)$, where F_i is the i^{th} matrix of F. Let check whether or not M_i/M_{i+1} is 0 or i-pure for $i = 0, 1, 2$.

```
> map(IsPure,F,A);
```

$$[\infty, 1, 2]$$

Let us now compute a new presentation of the left A-module $M = A/(A^{1\times 2}\,R)$ based on the grade filtration of M.

```
> P:=PurePresentationOfTorsionSubmodule(R,A);
```

$$P := [\begin{bmatrix} d_1 + d_2 - x_1 - x_2 & 1 \\ 0 & d_1 \\ 0 & d_2 \end{bmatrix}, \begin{bmatrix} {d_1}^2 + d_2 d_1 - (x_1 + x_2)\,d_1 - 1 \\ {d_2}^2 + d_2 d_1 - (x_1 + x_2)\,d_2 - 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -d_1 - d_2 + x_1 + x_2 \end{bmatrix}]$$

We get $L = A^{1\times 2}/(A^{1\times 3}\,P_1) \subseteq M = A/(A^{1\times 2}\,R)$. The injection $\iota : L \longrightarrow M$ is defined by $\iota(\kappa(\mu)) = \pi(\mu\,P_3)$, where $\kappa \colon A^{1\times 2} \longrightarrow L$ is the canonical projection onto L and P_i is the i^{th} matrix of P. Let us check again that ι is injective.

```
> TestInj(P[1],P[2],P[3],A);
```

$$true$$

Since $M = M_1 = t(M)$, ι is also an isomorphism, which can be easily check again.

```
> TestIso(P[1],P[2],P[3],A);
```

$$\mathit{true}$$

The inverse $\iota^{-1} : M \longrightarrow L$ of ι is then defined by

```
> T:=InverseMorphism(P[1],P[2],P[3],A);
```

$$T := [\begin{bmatrix} 1 & 0 \end{bmatrix}, \begin{bmatrix} d_1 & -1 & 0 \\ d_2 & 0 & -1 \end{bmatrix}]$$

i.e., $\iota^{-1}(\pi(\lambda)) = \kappa(\lambda\, T_1)$ for all $\lambda \in A$. Let us check again that the A-homomorphism defined by T_1 is an isomorphism.

```
> TestIso(P[2],P[1],T[1],A);
```

$$\mathit{true}$$

Let us now try to solve the linear PD system $P_1\, z = 0$ defined by the following PD equations:

```
> eqs:=map(a->a=0,convert(convert(ApplyMatrix(P[1],[zeta[1](iv),zeta[2](iv)],
> A),vector),set));
```

$$\mathit{eqs} :=$$

$$\Big\{ -\zeta_1(x_1, x_2)\, x_1 - \zeta_1(x_1, x_2)\, x_2 + \tfrac{\partial}{\partial x_1}\zeta_1(x_1, x_2) + \tfrac{\partial}{\partial x_2}\zeta_1(x_1, x_2) + \zeta_2(x_1, x_2) = 0,$$
$$\tfrac{\partial}{\partial x_1}\zeta_2(x_1, x_2) = 0,\ \tfrac{\partial}{\partial x_2}\zeta_2(x_1, x_2) = 0 \Big\}$$

We obtain:

```
> z:=pdsolve(eqs,zeta[1](iv),zeta[2](iv));
```

$$z :=$$

$$\Big\{ \zeta_1(x_1, x_2) = -1/2\, \Big(\_C1\, \sqrt{\pi} e^{1/4\,(-x_1 + x_2)^2} \mathrm{erf}\,(1/2\, x_1 + 1/2\, x_2) - 2\, \_F1\,(-x_1 + x_2) \Big)\, e^{{x_1}^2 + (-x_1 + x_2) x_1},$$
$$\zeta_2(x_1, x_2) = \_C1 \Big\}$$

In other words, the vector Z defined by

```
> Z:=evalm([[rhs(sol[1])],[rhs(sol[2])]]);
```

$$Z := \begin{bmatrix} -1/2\, \Big(\_C1\, \sqrt{\pi} e^{1/4\,(-x_1 + x_2)^2} \mathrm{erf}\,(1/2\, x_1 + 1/2\, x_2) - 2\, \_F1\,(-x_1 + x_2) \Big)\, e^{{x_1}^2 + (-x_1 + x_2) x_1} \\ \_C1 \end{bmatrix}$$

is a solution of the linear PD system $P_1\, Z = 0$.

```
> ApplyMatrix(P[1],Z,A);
```

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now, $y = T_1\, Z$, namely,

```
> y:=ApplyMatrix(T[1],Z,A);
```

$$y := \left[\begin{array}{c} -1/2\, \left(\_C1\, \sqrt{\pi} \mathrm{e}^{1/4\, (x_1 - x_2)^2} \mathrm{erf}\,(1/2\, x_1 + 1/2\, x_2) - 2\, \_F1\, (-x_1 + x_2) \right) \mathrm{e}^{x_1 x_2} \end{array} \right]$$

is a solution of the linear PD system $R\, y = 0$.

```
> ApplyMatrix(R,y,A);
```

$$\left[\begin{array}{c} 0 \\ 0 \end{array} \right]$$

We can directly integrate $P_1\, z = 0$ using the command `IntegrationOfTorsionDSubmodule`:

```
> U:=IntegrationOfTorsionDSubmodule(R,A);
```

$$U := \left[\begin{array}{c} -1/2\, \left(\_C1\, \sqrt{\pi} \mathrm{e}^{1/4\, (-x_1 + x_2)^2} \mathrm{erf}\,(1/2\, x_1 + 1/2\, x_2) - 2\, \_F1\, (-x_1 + x_2) \right) \mathrm{e}^{{x_1}^2 + (-x_1 + x_2) x_1} \\ \_C1 \end{array} \right]$$

```
> ApplyMatrix(P[1],U,A);
```

$$\left[\begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right]$$

Finally, the linear PD system $R\, y = 0$ can also be directly integrated using the command `IntegrationOfDModule`.

```
> X:=:=IntegrationOfDModule(R,A,a);
```

$$X := \left[\begin{array}{c} -1/2\, \left(\_C1\, \sqrt{\pi} \mathrm{e}^{1/4\, (x_1 - x_2)^2} \mathrm{erf}\,(1/2\, x_1 + 1/2\, x_2) - 2\, \_F1\, (-x_1 + x_2) \right) \mathrm{e}^{x_1 x_2} \end{array} \right]$$

```
> ApplyMatrix(R,X,A);
```

$$\left[\begin{array}{c} 0 \\ 0 \end{array} \right]$$

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